

Formal Synthesis of Control Strategies for Positive Monotone Systems

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Abstract

We design controllers from formal specifications for positive discrete-time monotone systems that are subject to bounded disturbances. Such systems are widely used to model the dynamics of transportation and biological networks. The specifications are described using signal temporal logic (STL), which can express a broad range of temporal properties. We formulate the problem as a mixed-integer linear program (MILP) and show that under the assumptions made in this paper, which are not restrictive for traffic applications, the existence of open-loop control policies is sufficient and almost necessary to ensure the satisfaction of STL formulas. We establish a relation between satisfaction of STL formulas in infinite time and set-invariance theories and provide an efficient method to compute robust control invariant sets in high dimensions. We also develop a robust model predictive framework to plan controls optimally while guaranteeing the satisfaction of the specification. Illustrative examples and a traffic management case study are included.

Index Terms

Formal Synthesis and Verification, Monotone Systems, Transportation Networks.

I. INTRODUCTION

In recent years, there has been a growing interest in using formal methods for specification, verification, and synthesis in control theory. Temporal logics [1] provide a rich, expressive framework for describing a broad range of properties such as safety, liveness, and reactivity. In formal synthesis, the goal is to control a dynamical system from a such a specification. For example, in an urban traffic network, a synthesis problem can be to generate traffic light control

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policies that ensure gridlock avoidance and fast enough traffic through a certain road, for all times.

Control synthesis for linear and piecewise affine systems from linear temporal logic (LTL) specifications was studied in [2]–[5]. The automata-based approach used in these works requires constructing finite abstractions that (bi)simulate the original system. Approximate finite bisimulation quotients for nonlinear systems were investigated in [6]–[8]. The main limitations of finite abstraction approaches are the large computational burden of discretization in high dimensions and conservativeness when exact bisimulations are impossible or difficult to construct. As an alternative approach, LTL optimization-based control of mixed-logical dynamical (MLD) systems [9] using mixed-integer programs was introduced in [10], [11], and was recently extended to model predictive control (MPC) from signal temporal logic (STL) specifications in [12]–[14]. However, these approaches are unable to guarantee infinite-time safety and the results are fragile in the presence of disturbances.

In some applications, the structural properties of the system and the specification can be exploited to consider alternative approaches to formal control synthesis. We are interested in systems in which the evolution of the state exhibits a type of *order preserving law* known as *monotonicity*, which is common in models of transportation, biological, and economic systems [15]–[19]. Such systems are also *positive* in the sense that the state components are always non-negative. Control of positive systems have been widely studied in the literature [20]–[22]. Positive linear systems are always monotone [21].

In this paper, we study optimal STL control of discrete-time positive monotone systems (i.e., systems with state partial order on the positive orthant) with bounded disturbances. STL allows designating time intervals for temporal operators, which makes it suitable for describing requirements with deadlines. Moreover, STL is equipped with quantitative semantics, which provides a measure to quantify how strongly the specification is satisfied/violated. The quantitative semantics of STL can also be used as cost for maximization in an optimal control setting. The STL specifications in this paper are restricted to a particular form that favors smaller values for the state components. We assume that there exists a maximal disturbance element that characterizes a type of upper-bound for the evolution of the system. These assumptions are specifically motivated by the dynamics of traffic networks, where the disturbances represent the volume of exogenous vehicles entering the network and the maximal disturbance characterizes the rush hour exogenous flow. Our optimal control study is focused on STL formulae with infinite-time safety/persistence

properties, which is relevant to optimal and correct traffic control in the sense that the vehicular flow is always free of congestion while the associated delay is minimized.

The key contributions of this paper are as follows. First, for finite-time semantics, we prove that the existence of open-loop control policies is necessary and sufficient for maintaining STL correctness. For the correctness of infinite-time semantics, we show that the existence of open-loop control sequences is sufficient and almost necessary, in a sense that is made clear in the paper. Implementing open-loop control policies is very simple since online state measurements are not required, which can prove useful in applications where the state is difficult to access. We use a robust MPC approach to optimal control. The main contribution of our MPC framework is guaranteed recursive feasibility, a property that was not established in prior STL MPC works [12]–[14]. We show via a case study that our method is applicable to systems with relatively high dimensions.

This remainder of the paper is organized as follows. We introduce the necessary notation and background on STL in Sec. II. The problems are formulated in Sec. III. The technical details for control synthesis from finite and infinite-time specifications are given in Sec. IV and Sec. V, respectively. The robust MPC framework is explained in Sec. VI. Finally, we introduce a traffic network model and explain its monotonicity properties in Sec. VII, where a case study is presented.

Related Work

This paper is an extension of our conference paper [23], where we studied safety control of positive monotone systems. Here, we significantly enrich the range of specifications to STL, provide complete proofs, and also include optimal control.

Monotone dynamical systems have been extensively investigated in the mathematics literature [24]–[27]. Early studies mainly focused on stability properties and characterization of limit sets for autonomous, deterministic continuous-time systems [24], [25], [28]. The results do not generally hold for discrete-time systems, as discussed in [26]. In particular, attractive periodic orbits are proven to be non-existent for continuous-time autonomous systems [28], but may exist for discrete-time autonomous systems. Here we present a similar result for controlled systems, where we show that a type of attractive periodic orbit exists for certain control policies.

Angeli and Sontag [29] extended the notion of monotonicity to deterministic continuous-time control systems and provided results on interconnections of these systems. However, they

assumed monotonicity with respect to both state and controls. In this paper, we do not require monotonicity with respect to controls, which enables us considering a broader class of systems. In particular, we do not require controls to belong to a partially ordered set.

This work is related to the literature on stabilization of switched positive linear systems. Stabilization requirements are closely related to set-invariance properties, which are thoroughly studied in this paper. The authors in [30], [31] investigated switching policies that result in exponential stabilization. Apart from considering richer specifications, we are able to accommodate significantly more complex systems. In particular, we are able to consider hybrid systems in which modes can be either determined directly by a control input or internally by the state.

Recently, there has been some work on formal verification and synthesis for monotone systems. Safety control of cooperative systems was investigated in [32]–[34]. However, these works, like [29], assumed monotonicity with respect to the control inputs as well. Some computational benefits gained from monotonicity were highlighted in [35] for reachability analysis of monotone hybrid systems. More recently, the authors in [36] provided an efficient method to compute finite abstractions for mixed-monotone systems (a more general class than monotone systems). While this approach can consider systems and specifications beyond the assumptions in this paper, it still requires state-space discretization, which is a severe limitation in high dimensions. Moreover, this method is conservative, since the finite abstractions are often not bisimilar with the original system. In contrast, our approach provides a notion of (approximate) completeness.

II. PRELIMINARIES

A. Notation

For two integers a, b , we use $\text{rem}(a, b)$ to denote the remainder of division of a by b (a modulo b). Given a set \mathcal{S} and a positive integer K , we use the shorthand notation \mathcal{S}^K for $\prod_{i=1}^K \mathcal{S}$. A discrete-time signal is defined as an infinite sequence $\mathbf{s} = s_0 s_1 \cdots$, where $s_k \in \mathcal{S}$, $k \in \mathbb{N}$. Given $s \in \mathcal{S}$, the infinite-sequence of repetitions of s , $ss \cdots$, is denoted by $(s)^\omega$. The set of all signals that can be generated from \mathcal{S} is denoted by \mathcal{S}^ω . We use $\mathbf{s}[k] = s_k s_{k+1} \cdots$ and $\mathbf{s}[k_1 : k_2] = s_{k_1} s_{k_1+1} \cdots s_{k_2}$, $k_1 < k_2$, to denote specific portions of \mathbf{s} . A discrete-time *real* signal is $\mathbf{r} = r_0 r_1 r_2 \cdots$, where $r_k \in \mathbb{R}^n, \forall k \in \mathbb{N}$.

We denote the positive closed orthant of the n -dimensional Euclidian space by

$$\mathbb{R}_+^n := \{x \in \mathbb{R}^n | x_{[i]} \geq 0, i = 1, \dots, n\},$$

where $x = (x_{[1]}, x_{[2]}, \dots, x_{[n]})^T$. We denote the vector of all ones in \mathbb{R}^n by 1_n . For two vectors $a, b \in \mathbb{R}^n$, the non-strict partial order relation \preceq is defined as: $a \preceq b \Leftrightarrow b - a \in \mathbb{R}_+^n$. For $x \in \mathbb{R}_+^n$, let $L(x) = \{y \in \mathbb{R}_+^n \mid y \preceq x\}$.

Definition 1 ([37]). A set $\mathcal{X} \subset \mathbb{R}_+^n$ is a *lower-set* if $L(x) \subseteq \mathcal{X}$, $\forall x \in \mathcal{X}$.

It is straightforward to verify that if $\mathcal{X}_1, \mathcal{X}_2$ are lower-sets, then $\mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathcal{X}_1 \cap \mathcal{X}_2$ are also lower-sets.

We extend the usage of notation \preceq to equal-length real signals: for two real signals \mathbf{r}, \mathbf{r}' , we denote $\mathbf{r}'[t'_1 : t'_2] \preceq \mathbf{r}[t_1 : t_2]$, $t_2 - t_1 = t'_2 - t'_1$, if $r'_{t'_1+k} \preceq r_{t_1+k}$, $k = 0, 1, \dots, t_2 - t_1$.

B. Signal Temporal Logic (STL)

In this paper, STL [38] formulas are defined over discrete-time real signals. The syntax of negation-free STL is:

$$\varphi := \mu \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathbf{U}_I \varphi_2 \mid \mathbf{F}_I \varphi \mid \mathbf{G}_I \varphi,$$

where

$$\mu := (p(r) \leq c)$$

is a predicate on $r \in \mathbb{R}^n$, $p : \mathbb{R}^n \rightarrow \mathbb{R}$, $c \in \mathbb{R}$; \wedge and \vee are Boolean connectives for conjunction and disjunction, respectively; \mathbf{U}_I , \mathbf{F}_I , \mathbf{G}_I are the timed *until*, *eventually* and *always* operators, respectively, and $I = [t_1, t_2]$ is a time interval, $t_1, t_2 \in \mathbb{N} \cup \{\infty\}$, $t_2 \geq t_1$. For the case $t_1 = t_2$, we use the shorthand notation $\{t_1\} := [t_1, t_1]$. Exclusion of negation does not restrict expressivity of temporal properties. It can be easily shown that any temporal logic formula can be brought into *negation normal form* (where all negation operators apply to the predicates) [14], [39]. We deliberately omit negation from STL syntax for laying out properties that are later exploited in the paper. For simplicity, in the rest of the paper, we will refer to negation-free STL simply as STL. The semantics of STL is inductively defined as:

$$\begin{aligned} \mathbf{r}[t] \models \mu & \Leftrightarrow p(r) \leq c, \\ \mathbf{r}[t] \models \varphi_1 \vee \varphi_2 & \Leftrightarrow \mathbf{r}[t] \models \varphi_1 \vee \mathbf{r}[t] \models \varphi_2, \\ \mathbf{r}[t] \models \varphi_1 \wedge \varphi_2 & \Leftrightarrow \mathbf{r}[t] \models \varphi_1 \wedge \mathbf{r}[t] \models \varphi_2, \\ \mathbf{r}[t] \models \varphi_1 \mathbf{U}_I \varphi_2 & \Leftrightarrow \exists t' \in t + I \text{ s.t. } \mathbf{r}[t'] \models \varphi_2 \\ & \quad \wedge \forall t'' \in [t, t'], \mathbf{r}[t''] \models \varphi_1, \\ \mathbf{r}[t] \models \mathbf{F}_I \varphi & \Leftrightarrow \exists t' \in t + I \text{ s.t. } \mathbf{r}[t'] \models \varphi, \\ \mathbf{r}[t] \models \mathbf{G}_I \varphi & \Leftrightarrow \forall t' \in t + I \text{ s.t. } \mathbf{r}[t'] \models \varphi, \end{aligned} \tag{1}$$

where \models is read as *satisfies*. The *language* of φ is the set of all signals such that $\mathbf{r}[0] \models \varphi$. The *horizon* of an STL formula φ , denoted by h^φ , is defined as the time required to decide the satisfaction of φ , which is recursively computed as [40]:

$$\begin{aligned} h^\mu &= 0, \\ h^{\varphi_1 \wedge \varphi_2} &= h^{\varphi_1 \vee \varphi_2} = \max(h^{\varphi_1}, h^{\varphi_2}), \\ h^{\mathbf{F}_{[t_1, t_2]} \varphi} &= h^{\mathbf{G}_{[t_1, t_2]} \varphi} = t_2 + h^\varphi, \\ h^{\varphi_1 \mathbf{U}_{[t_1, t_2]} \varphi_2} &= t_2 + \max(h^{\varphi_1}, h^{\varphi_2}). \end{aligned} \tag{2}$$

Definition 2. An STL formula φ is *bounded* if $h^\varphi < \infty$.

The satisfaction of φ by $\mathbf{r}[t]$ is decided only by $\mathbf{r}[t : t + h^\varphi]$ and the rest of the signal valuations are irrelevant. Therefore, instead of $\mathbf{r}[t] \models \varphi$, we occasionally write $\mathbf{r}[t : t + h^\varphi] \models \varphi$ with the same meaning.

The *STL robustness score* $\rho(\mathbf{r}, \varphi, t) \in \mathbb{R}$ is a measure indicating how strongly φ is satisfied by $\mathbf{r}[t]$, which is recursively computed as [38]:

$$\begin{aligned} \rho(\mathbf{r}, \mu, t) &= c - p(r), \\ \rho(\mathbf{r}, \varphi_1 \vee \varphi_2, t) &= \max(\rho(\mathbf{r}, \varphi_1, t), \rho(\mathbf{r}, \varphi_2, t)), \\ \rho(\mathbf{r}, \varphi_1 \wedge \varphi_2, t) &= \min(\rho(\mathbf{r}, \varphi_1, t), \rho(\mathbf{r}, \varphi_2, t)), \\ \rho(\mathbf{r}, \varphi_1 \mathbf{U}_I \varphi_2, t) &= \max_{t' \in t+I} \left(\min(\rho(\mathbf{r}, \varphi_1, t'), \right. \\ &\quad \left. \min_{t'' \in [t, t']} \rho(\mathbf{r}, \varphi_2, t'')) \right), \\ \rho(\mathbf{r}, \mathbf{F}_I \varphi, t) &= \max_{t' \in I} \rho(\mathbf{r}, \varphi, t'), \\ \rho(\mathbf{r}, \mathbf{G}_I \varphi, t) &= \min_{t' \in I} \rho(\mathbf{r}, \varphi, t'). \end{aligned} \tag{3}$$

Positive (respectively, negative) robustness indicates satisfaction (respectively, violation) of the formula.

Example 1. Consider a one-dimensional signal \mathbf{r} , where $r_k = k, k \in \mathbb{N}$, and the predicate $\mu = (r^2 \leq 10)$. We have $\rho(\mathbf{r}, \mathbf{G}_{[0,3]} \mu, 0) = \min(10 - 0, 10 - 1, 10 - 4, 10 - 9) = 1$ (satisfaction) and $\rho(\mathbf{r}, \mathbf{F}_{[4,6]} \mu, 0) = \max(10 - 16, 10 - 25, 10 - 36) = -6$ (violation).

Remark 1. There are minor differences between the original STL introduced in [38] and the one used in this paper. In [38], STL was developed as an extension of metric interval temporal logic (MITL) [41] for real-valued continuous-time signals. Here, without any loss of generality, we apply STL to discrete-time signals. Our STL is based on metric temporal logic (MTL) (similar to [40]). Thus, we allow the intervals of temporal operators to be singletons (punctual)

or unbounded. It is worth to note that STL formulas in this paper can be translated into LTL by replacing the time intervals of temporal operators with appropriate nestings of LTL “next” operator. However, the LTL representation of STL formulas would be very inefficient. We prefer STL for convenience of specifying requirements for systems with real-valued states. We also exploit the STL quantitative semantics.

III. PROBLEM STATEMENT AND APPROACH

We consider discrete-time systems of the following form:

$$x_{t+1} = f(x_t, u_t, w_t), \quad (4)$$

where $x_t \in \mathcal{X}$ is the state, $\mathcal{X} \subset \mathbb{R}_+^n$, $u_t \in \mathcal{U}$ is the control input, $\mathcal{U} = \mathbb{R}^{m_r} \times \{0, 1\}^{m_b}$, and $w_t \in \mathcal{W}$ is the disturbance (adversarial input) at time t , $t \in \mathbb{N}$, $\mathcal{W} = \mathbb{R}^{q_r} \times \{0, 1\}^{q_b}$. The sets \mathcal{U} and \mathcal{W} may include real and binary values. For instance, the set of controls in the traffic model developed in Sec. VII includes binary values for decisions on traffic lights and real values for ramp meters. These types of systems are positive as all state components are non-negative. We also assume that \mathcal{X} is bounded.

Definition 3. System (4) is monotone (with partial order on \mathbb{R}_+^n) if for all $x, x' \in \mathcal{X}$, $x' \preceq x$, we have $f(x', u, w) \preceq f(x, u, w)$, $\forall u \in \mathcal{U}, \forall w \in \mathcal{W}$.

The systems considered in this paper are positive and monotone with partial order on \mathbb{R}_+^n . For the remainder of the paper, we simply refer to systems in Definition 3 as monotone¹. Although the results of this paper are valid for any general $f : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{X}$, we focus on systems that can be written in the form of mixed-logical dynamical (MLD) systems [9], which are defined in Sec. IV. It is well known that a wide range of systems involving discontinuities (hybrid systems), such as piecewise affine systems, can be transformed into MLDs [42].

Assumption 1. There exist $w^* \in \mathcal{W}$ such that

$$\forall x \in \mathcal{X}, \forall u \in \mathcal{U}, f(x, u, w) \preceq f(x, u, w^*), \forall w \in \mathcal{W}. \quad (5)$$

We denote $f(x, u, w^*)$ by $f^*(x, u)$ and refer to f^* as the *maximal system*. As it will be further explained in this paper, the behavior of monotone system (4) is mainly characterized

¹In dynamic systems theory, the term *cooperative* is specifically used for referring to systems that are monotone with partial order on the positive orthant. However, we avoid using this term for control systems as it might generate confusion with the similar terminology used in the literature on multi-agent control systems.

by its maximal f^* . The assumption above is restrictive but it holds for many compartmental systems where the disturbances are additive and the components are independent. Therefore, the maximal system corresponds to the situation where each component takes its most extreme value. We also note that if Assumption 1 is removed, overestimating f by some f^* such that $f(x, u, w) \preceq f^*(x, u), \forall w \in \mathcal{W}$, is always possible for a bounded f . By overestimating f the control synthesis methods of this paper remain correct, but become conservative.

We describe the desired system behavior using specifications written as STL formulas over a finite set of predicates on the state. We assume that each predicate μ is in the following form:

$$\mu = (a_\mu^T x \leq b_\mu), \quad (6)$$

where $a_\mu \in \mathbb{R}_+^n$, $b_\mu \in \mathbb{R}_+$. It is straightforward to verify that the closed half-space defined by (6) is a lower-set in \mathbb{R}_+^n . By restricting the predicates into the form (6), we ensure that a predicate remains true if the values of state components are decreased. This restriction is motivated by monotonicity. For example, in a traffic network, the state is the vector representation of vehicular densities in different segments of the network. The satisfaction of a “sensible” traffic specification has to be preserved if the vehicular densities are not increased all over the network. Otherwise, the specification encourages large densities and congestion.

Definition 4. A control policy $\mu := \bigcup_{t \in \mathbb{N}} \mu_t$ is a set of functions $\mu_t : \mathcal{X}^{t+1} \rightarrow \mathcal{U}$, where

$$u_t = \mu_t(x_0, x_1, \dots, x_t).$$

An *open-loop* control policy takes the simpler form $u_t = \mu_t(x_0)$, i.e. the decision on the sequence of control inputs is made using only the initial state x_0 and not the subsequent state measurements. On the other hand, in a (history dependent) *feedback* control policy, $u_t = \mu_t(x_0, x_1, \dots, x_t)$, the controller implementation requires real-time access to the state and its history.

Following the notation introduced in Sec. II-A, the set of all infinite-length sequences of admissible disturbances is denoted by \mathcal{W}^ω , where each $\mathbf{w} \in \mathcal{W}^\omega$ is $\mathbf{w} = w_0 w_1 \dots$, $w_k \in \mathcal{W}$, $k \in \mathbb{N}$. Given an initial condition x_0 , a control policy μ and $\mathbf{w} \in \mathcal{W}^\omega$, the *run* of the system is defined as the following signal:

$$\mathbf{x} = \mathbf{x}(x_0, \mu, \mathbf{w}) := x_0 x_1 x_2 \dots,$$

where $x_{t+1} = f(x_t, u_t, w_t), \forall t \in \mathbb{N}$. Now we formulate the problems studied in this paper. In all problems, we assume a monotone system (4) is given, Assumption 1 holds, and all the predicates are in the form of (6).

Problem 1 (Bounded STL Control). Given a bounded STL formula φ , find a set of initial conditions $\mathcal{X}_0 \subset \mathcal{X}$ and a control policy μ such that

$$\mathbf{x}(x_0, \mu, \mathbf{w})[0] \models \varphi, \forall \mathbf{w} \in \mathcal{W}^\omega, \forall x_0 \in \mathcal{X}_0.$$

As mentioned in the previous section, the satisfaction of a bounded φ solely depends on $\mathbf{x}[0 : h^\varphi]$, where horizon h^φ is obtained from (2). The formula horizon h^φ can be viewed as the time when the specification ends. In many engineering applications, guaranteeing infinite-time properties is important. These specifications are usually expressed as safety/persistence properties, which indicate that the system is required to uphold certain behaviors for all times. We formulate these specifications as *global* STL formulas in the form of

$$\mathbf{G}_{[0, \infty]} \varphi, \tag{7}$$

where φ is a bounded STL formula and $\mathbf{G}_{[0, \infty]}$ stands for unbounded temporal “always”, as defined in Sec. II-B.

Problem 2 (Global STL Control). Given a bounded STL formula φ , find a set of initial conditions $\mathcal{X}_0 \subset \mathcal{X}$ and a control policy μ such that

$$\mathbf{x}(x_0, \mu, \mathbf{w})[0] \models \mathbf{G}_{[0, \infty]} \varphi, \forall \mathbf{w} \in \mathcal{W}^\omega, \forall x_0 \in \mathcal{X}_0.$$

Remark 2. A finite-length signal can satisfy an STL formula that does not contain any unbounded “always” operator. Therefore, we do not require separate problem formulations for STL formulas containing unbounded “eventually” or “until” operators as their unbounded time intervals can be safely under-approximated by bounded intervals. However, bounded under-approximation is not sound for the unbounded “always” operator, as in (7). A global formula can only be satisfied (respectively, violated) with infinite-length (respectively, finite-length) signals. Therefore, we have distinguished Problems 1 and Problem 2 as the approaches to their solutions belong to different paradigms. We refer the interested reader to a detailed discussion on decidability of MTL formulas in [41].

In the presence of disturbances, feedback controllers obviously outperform open-loop controllers. We show that the existence of open-loop control policies for guaranteeing the STL correctness of monotone systems in Problem 1 (respectively, Problem 2) is sufficient and (respectively, almost) necessary.

While, as we will show later in the solution to Problem 1, the online knowledge of state is not necessary for STL correctness, it can be exploited for planning controls optimally. In this paper, we study the robust optimal control problem for global formulas (Equ. 7), which is of practical interest for optimal traffic management (as discussed in Sec. VII). We use a model predictive control (MPC) approach, which is a popular, powerful approach to optimal control of constrained systems. Given a planning horizon of length H ², a sequence of control actions starting from time t is denoted by $u_t^H := u_{0|t}, u_{1|t}, \dots, u_{H-1|t}$. Given u_t^H and x_t , we denote the predicted H -step system response by

$$x_t^H(x_t, u_t^H, w_t^H) := x_{1|t}, x_{2|t}, \dots, x_{H|t},$$

where $x_{k+1|t} = f(x_{k|t}, u_{k|t}, w_{k|t})$, $k = 0, 1, \dots, H-1$, $x_{0|t} = x_t$ and $w_t^H := w_{0|t}, w_{1|t}, \dots, w_{H-1|t}$. At each time, u_t^H is found such that it optimizes a cost function $J(x_t^H, u_t^H)$, $J : \mathcal{X}^H \times \mathcal{U}^H \rightarrow \mathbb{R}$, subject to system constraints. When u_t^H is computed, only the first control action $u_{0|t}$ is applied to the system and given the next state, the optimization problem is resolved for u_{t+1}^H . Therefore, the implementation has a closed-loop behavior.

Problem 3 (Robust STL MPC). Given a bounded STL formula φ , an initial condition x_0 , a planning horizon H and a cost function $J(x_t^H, u_t^H)$, find a control policy μ such that

$$\begin{aligned} u_t = & \operatorname{argmin}_{\forall w_t^H} \max J(x_t^H(x_t, u_t^H, w_t^H), u_t^H), \\ \text{s.t. } & \mathbf{x}(x_0, \mu, \mathbf{w})[0] \models \mathbf{G}_{[0, \infty]} \varphi, \forall \mathbf{w} \in \mathcal{W}^\omega, \\ & x_{k+1} = f(x_k, u_k, w_k), \forall k \in \mathbb{N} \end{aligned} \quad (8)$$

For computational purposes, we assume that J is a piecewise affine function of the state and controls. Moreover, the cost functions in our applications are non-decreasing with respect to the state in the sense that $x'_{k|t} \preceq x_{k|t}$, $k = 1, 2, \dots, H \Rightarrow J(x_t^{H'}, u_t^H) \preceq J(x_t^H, u_t^H)$, $\forall u_t^H \in \mathcal{U}^H$. As it will become clear later in the paper, we will exploit this property to simplify the worst-case optimization problem in (8) to an optimization problem for the maximal system.

²The MPC horizon H should not be confused with the STL horizon h^φ .

As mentioned earlier, a natural objective is maximizing STL robustness score. It follows from the linearity of the predicates in (6) and max and min operators in (3) that STL robustness score is a piecewise affine function of finite-length signals. We can also consider optimizing a weighted combination of STL robustness score and a given cost function. We use this cost formulation in the traffic application in Sec. VII.

The primary challenge of robust STL MPC is guaranteeing the satisfaction of the global STL formula while the controls are planned in a receding horizon manner (see the constraints in (8)). Our approach takes the advantage of the results from Problem 2 to design appropriate terminal sets for the MPC algorithm such that the generated runs are guaranteed to satisfy the global STL specification while the online control decisions are computed (sub)optimally. Due to the temporal logic constraints, our MPC setup differs from the conventional one. The details are explained in Sec. VI.

IV. FINITE HORIZON SEMANTICS

In this section, we explain the solution to Problem 1. First, we exploit monotonicity to characterize the properties of the solutions. Next, we explain how to synthesize controls using a mixed integer linear programming (MILP) solver.

Lemma 1. Consider runs \mathbf{x} and \mathbf{x}' and an STL formula φ . If for some t, t' , we have $\mathbf{x}'[t : t + h^\varphi] \preceq \mathbf{x}[t' : t' + h^\varphi]$, then $\mathbf{x}[t] \models \varphi$ implies $\mathbf{x}'[t'] \models \varphi$.

Proof: Since all predicates denote lower-sets in the form of (6), then $x'_{t'} \preceq x_t$ implies $a_\mu^T x'_{t'} \leq a_\mu^T x_t$. Thus $\mathbf{x}[t] \models \mu$ implies $\mathbf{x}'[t'] \models \mu$. Hence all predicates that were true by the valuations in \mathbf{x} remain true for \mathbf{x}' . It follows from the negation-free semantics in (1) that without falsifying a predicate, a formula can not be falsified. Therefore $\mathbf{x}[t] \models \varphi$ implies $\mathbf{x}'[t'] \models \varphi$ ■

The *largest set of admissible initial conditions* is defined as:

$$\mathcal{X}_0^{\max} := \left\{ x_0 \in \mathcal{X} \mid \exists \mu \text{ s.t. } \mathbf{x}(x_0, \mu, \mathbf{w}) \models \varphi, \forall \mathbf{w} \in \mathcal{W}^\omega \right\}.$$

The set \mathcal{X}_0^{\max} is a union of polyhedra. Finding the half-space representation of all polyhedral sets in \mathcal{X}_0^{\max} may not be possible for high dimensions. Therefore, we find a half-space representation for a subset of \mathcal{X}_0^{\max} . The following result states how to check whether $x_0 \in \mathcal{X}_0^{\max}$.

Theorem 1. We have $x_0 \in \mathcal{X}_0^{\max}$ if and only if there exists an open-loop control sequence

$$u_0^{ol, x_0} u_1^{ol, x_0} \dots u_{h^\varphi-1}^{ol, x_0}$$

such that $\mathbf{x}^{ol,x_0}[0 : h^\varphi] \models \varphi$, where $\mathbf{x}^{ol,x_0}[0 : h^\varphi] = x_0^{ol,x_0} x_1^{ol} \cdots x_{h^\varphi}^{ol,x_0}$, and $x_{k+1}^{ol,x_0} = f^*(x_k^{ol,x_0}, u_k^{ol,x_0})$, $k = 0, \dots, h^\varphi - 1$, $x_0^{ol,x_0} = x_0$.

Proof: (Necessity) Consider system (4) such that $w_k = w^*$, $k = 0, 1, \dots, h^\varphi - 1$. Recall from Assumption 1 that $w^* \in \mathcal{W}$. Thus, such a \mathbf{w} is a valid disturbance sequence. Satisfaction of φ with $w_k = w^*$ requires at least one satisfying run for the maximal system, hence a corresponding control sequence by $u_0^{ol,x_0}, u_1^{ol}, \dots, u_{h^\varphi-1}^{ol,x_0}$. *(Sufficiency)* Consider the run generated by the original system $x_{k+1} = f(x_k, u_k^{ol,x_0}, w_k)$. We prove that $x_k \preceq x_k^{ol,x_0}$, $k = 0, 1, \dots, h^\varphi$, by induction over k . The base case $x_0 \preceq x_0^{ol,x_0}$ is trivial ($x_0 = x_0^{ol,x_0}$). The inductive step is verified from monotonicity: $x_{k+1} = f(x_k, u_k^{ol,x_0}, w_k) \preceq f^*(x_k^k, u_k^k) = x_{k+1}^{ol,x_0}$. Therefore, $\mathbf{x}[0 : h^\varphi] \preceq \mathbf{x}^{ol,x_0}[0 : h^\varphi]$, $\forall \mathbf{w}[0 : h^\varphi] \in \mathcal{W}^{h^\varphi}$. It follows from Lemma 1 that $\mathbf{x}[0 : h^\varphi] \models \varphi$, $\forall \mathbf{w}[0 : h^\varphi] \in \mathcal{W}^{h^\varphi}$. ■

Corollary 1. The set \mathcal{X}_0^{max} is a lower-set.

Proof: Consider any $x'_0 \in L(x_0)$. Let $x'_{k+1}, f(x'_k, u_k^{ol,x_0}, w_k)$, $k = 0, 1, \dots, h^\varphi - 1$. It follows from monotonicity that $x'_k \preceq x_k^{ol,x_0}$, $k = 0, 1, \dots, h^\varphi$, $\forall \mathbf{w}[0 : h^\varphi] \in \mathcal{W}^{h^\varphi}$. By the virtue of Lemma 1, $\mathbf{x}'[0 : h^\varphi] \preceq \mathbf{x}^{x_0,ol}[0 : h^\varphi]$. Therefore, $\forall x_0 \in \mathcal{X}_0^{max}$, we have $x'_0 \in \mathcal{X}_0^{max}$, $\forall x'_0 \in L(x_0)$, which indicates \mathcal{X}_0^{max} is a lower-set. ■

Corollary 2. If $x_0 \in \mathcal{X}_0^{max}$ and μ^{ol} is the following open-loop control policy

$$\mu_t^{ol}(x_0) = u_t^{ol,x_0}, t = 0, 1, \dots, h^\varphi - 1,$$

then $\mathbf{x}(x_0, \mu, \mathbf{w})[0 : h^\varphi] \models \varphi$, $\forall \mathbf{w} \in \mathcal{W}^{h^\varphi}$, $\forall x_0 \in L(x_0)$.

Proof: Follows from the proof of Corollary 1. ■

Now that we have established the properties of the solutions to Problem 1, we explain how to compute the admissible initial conditions and their corresponding open-loop control sequences. The approach is based on formulating the conditions in Theorem 1 as a set of constraints that can be incorporated into a feasibility solver. We convert all the constraints into a set of mixed-integer linear constraints and use off-the-shelf MILP solvers to check for feasibility. Converting logical properties into mixed-integer constraints is a common procedure which was employed for MLD systems in [9]. The authors in [10] and [12] extended this technique to a framework for time bounded model checking of temporal logic formulas. A variation of this method is explained here.

First, the STL formula is recursively translated into a set of mixed-integer constraints. For each predicate $\mu = (a_\mu^T x \leq b_\mu)$, as in (6), we define a binary variable $z_k^\mu \in \{0, 1\}$ such that 1

(respectively, 0) stands for true (respectively, false). The relation between z^μ , robustness ρ , and x is encoded as:

$$a_\mu^T x - M(1 - z^\mu) + \rho \leq b_\mu, \quad (9a)$$

$$a_\mu^T x + Mz^\mu + \rho \geq b_\mu. \quad (9b)$$

The constant M is a sufficiently large number such that $M \geq \max\{a_\mu^T K, b_\mu\}$, where $K \in \mathbb{R}_+^n$ is the upper bound for the state values, $x_k \preceq K, \forall k \in \{0, 1, \dots, h^\varphi\}$. In practice, M is chosen sufficiently large such that the constraint $x \preceq K$ is never active. Note that the largest value of ρ for which $z^\mu = 1$ is $b - a_\mu^T x$, which is equal to the robustness of μ .

Now we encode the truth table relations. For instance, we desire to capture $1 \wedge 0 = 0$ and $1 \vee 0 = 1$ using mixed-integer linear equations. The binary relations for disjunction and conjunction connectives are encoded as the following constraints:

$$z = \bigwedge_{i=1}^{n_z} z_i \Rightarrow z \leq z_i, i = 1, \dots, n_z, \quad (10a)$$

$$z = \bigvee_{i=1}^{n_z} z_i \Rightarrow z \geq \sum_{i=1}^{n_z} z_i, \quad (10b)$$

where $z \in [0, 1]$ is declared as a continuous variables. However, it only can take binary values as explained shortly. We similarly define $z_k^\varphi \in [0, 1]$ as the variable indicating whether $\mathbf{x}[k] \models \varphi$. An STL formula is recursively translated as:

$$\varphi = \bigwedge_{i=1}^{n_\varphi} \varphi_i \Rightarrow z_k^\varphi = \bigwedge_{i=1}^{n_\varphi} z_k^{\varphi_i}; \quad (11a)$$

$$\varphi = \bigvee_{i=1}^{n_\varphi} \varphi_i \Rightarrow z_k^\varphi = \bigvee_{i=1}^{n_\varphi} z_k^{\varphi_i}; \quad (11b)$$

$$\varphi = \mathbf{G}_I \psi \Rightarrow z_k^\varphi = \bigwedge_{k' \in I} z_{k'}^\psi; \quad (11c)$$

$$\varphi = \mathbf{F}_I \psi \Rightarrow z_k^\varphi = \bigvee_{k' \in I} z_{k'}^\psi; \quad (11d)$$

$$\varphi = \psi_1 \mathbf{U}_I \psi_2 \Rightarrow z_k^\varphi = \bigvee_{k' \in I} \left(z_{k'}^{\psi_2} \wedge \bigwedge_{k'' \in [k, k']} z_{k''}^{\psi_1} \right). \quad (11e)$$

Finally, we add the following constraints:

$$z_0^\varphi = 1, \rho \geq 0. \quad (12)$$

Proposition 1. The set of constraints in (9),(11),(10),(12) has the following properties:

- i) we have $\mathbf{x}[0] \models \varphi$ if the set of constraints is feasible;
- ii) we have $\mathbf{x}[0] \not\models \varphi$ if the set of constraints is infeasible;
- iii) the largest ρ such that the set of constraints - while “ $\rho \geq 0$ ” is removed from (12) - is feasible is equal to $\rho(\mathbf{x}, \varphi, 0)$.

Proof. i) We provide the proof for (10), as the case for more complex STL formulas are easily followed in a recursive manner. If $z = 1$, then it follows from (10a) that $z_i = 1, i = 1, \dots, n_z$, which correctly encodes conjunctions. Similarly, for (10b) we have $z = 1$ indicating that $\exists i \in \{1, \dots, n_z\}$ such that $z_i = 1$, which correctly encodes disjunctions. ii) Infeasibility can be recursively traced back into (10). For (10a), if $z = 1$ is infeasible, it indicates that $z_i = 0, i = 1, \dots, n_z$. Therefore, all corresponding predicates are violated. This correctly encodes violation of conjunctions. Similarly, for (10b) infeasibility of $z = 1$ indicates $z_i = 0, i = 1, \dots, n_z$, which correctly encodes violation of conjunctions. iii) We also prove this statement for (10) as it is the base of recursion for general STL formulas. Let $z_i = (a_{\mu_i}^T x + \rho \leq b_{\mu_i}, i = 1, \dots, n_z$. Consider (10a) and the following optimization problem:

$$\begin{aligned} \rho^{\max} &= \operatorname{argmax} \quad \rho, \\ \text{s.t.} \quad &a_{\mu_i}^T x + \rho \leq b_{\mu_i}, i = 1, \dots, n_z, \end{aligned}$$

where the solution is $\rho^{\max} = \min_{i=1, \dots, n_z} (b_{\mu_i} - a_{\mu_i}^T x)$, which is identical to the quantitative semantics for conjunction (see (3)). Similarly, consider (10b) and the following optimization problem:

$$\begin{aligned} \rho^{\max} &= \operatorname{argmax} \quad \rho, \\ \text{s.t.} \quad &\exists i \in \{1, \dots, n_z\}, a_{\mu_i}^T x + \rho \leq b_{\mu_i}, \end{aligned}$$

where the solution is $\rho^{\max} = \max_{i=1, \dots, n_z} (b_{\mu_i} - a_{\mu_i}^T x)$, which is identical to the quantitative semantics for disjunction. \square

Our integer formulation for Boolean connectives slightly differs from the formulation in [10], [12], where lower bound constraints for the z 's are required. For example, for translating $z = \bigwedge_{i=1}^{n_z} z_i$, it is required to add $z \geq \sum_{i=1}^{n_z} z_i - n_z + 1$ to impose a lower bound for z . However, these additional constraints become necessary only when the negation operator is present in the STL formula. Hence, they are removed in our formulation, which reduces the constraint redundancy and degeneracy of the problem. By doing so, we observed significant computation speed gains (up to half computation time) in our case studies. Moreover, we encode quantitative semantics

in a different way than [12], where a separate STL robustness-based encoding is developed. Our encoding does not introduce additional integers hence it is computationally more efficient.

Definition 5. System (4) is in MLD form [9] if it is written as:

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t + D_\delta \delta_t + D_r r_t, \quad (13a)$$

$$E_\delta \delta_t + E_r r_t \preceq E_x x_t + E_u u_t + E_w w_t + e, \quad (13b)$$

where $\delta_t \in \{0, 1\}^{n_\delta}$ and $r_t \in \mathbb{R}^{n_r}$ are auxiliary variables and $A, B_u, B_w, D_\delta, D_r, E_\delta, E_r, E_x, E_u, E_w, e$ are appropriately defined constant matrices such that (13) is well-posed in the sense that given x_t, u_t, w_t , the feasible set for x_{t+1} is a single point equal to $f(x_t, u_t, w_t)$.

The system equations are brought into mixed-integer linear constraints by transforming system (4) into the MLD form. As mentioned earlier, any piecewise affine system can be transformed into an MLD. In the case studies of this paper, the construction of (13) from a piecewise affine (4) is not explained as the procedure is well documented in [42].

Finally, the set of constraints in Theorem 1 can be cast as:

$$\begin{cases} x_0^{ol, x_0} = x_0, & \text{Initial condition;} \\ x_{k+1}^{ol, x_0} = f^*(x_k^{ol, x_0}, u_k^{ol, x_0}), & \text{System constraints;} \\ z_k^\mu = (a_\mu^T x_k^{ol, x_0} \leq b_\mu), & \text{Predicate evaluations;} \\ z_0^\varphi = 1, \rho \geq 0, & \text{STL satisfaction.} \end{cases} \quad (14)$$

Checking the satisfaction of the set of constraints in (14) can be formulated as a MILP feasibility problem, which is handled using powerful off-the-shelf solvers. For a fixed initial condition x_0 , the feasibility of the MILP indicates whether $x_0 \in \mathcal{X}_o^{\max}$. An explicit representation of \mathcal{X}_o^{\max} requires variable elimination from (14), which is computationally intractable for a large MILP. Alternatively, we can set x_0 as a free variable while maximizing a cost function (e.g. norm of x_0) such that a large $L(x_0)$ is obtained. Another natural candidate is maximizing $\rho(\mathbf{x}^{ol, x_0}, \varphi, 0)$. It is worth to note that by finding a set of distinct initial conditions and taking the union of all $L(x_0)$, we are able to find a representation for an under-approximation of \mathcal{X}_o^{\max} .

MILPs are NP-complete. The complexity of solving (14) grows exponentially with respect to the number of binary variables and polynomially with respect to the number of continuous variables. The number of binary variables in our framework is $\mathcal{O}(h^\varphi(|P| + m_b + q_b + n_\delta))$ and the number of continuous variables is $\mathcal{O}(h^\varphi(n + m_r + q_r + n_r))$. In other words, the exponential growth builds upon the intricacy of the specification and the number of modes demonstrated by

the hybrid nature of the system. However, the complexity is polynomial with respect to the dimension of the state.

Example 2. Consider the following switched system:

$$x^+ = e^{A_u \tau} x + A_u^{-1}(I - e^{-A_u \tau})w,$$

where $x = (x_{[1]}, x_{[2]})^T \in \mathbb{R}_+^2$, $u \in \mathcal{U}$ is the control input (switch), $\mathcal{U} = \{1, 2\}$, and

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}, A_2 = \begin{pmatrix} -8 & 1 \\ 1 & 2 \end{pmatrix}.$$

The (additive) disturbance w is bounded to $L(w^*)$, where $w^* = (1.5, 1)^T$ and $\tau = 0.1$. Note that this system is the discrete-time version of $\dot{x} = A_u x + w$ with sample time τ . Both matrices are Metzler (all off-diagonal terms are non-negative hence all the elements of its exponential are positive) and non-Hurwitz hence constant input results in unbounded trajectories. The system is desired to satisfy the following STL formula:

$$\varphi = \bigvee_{T=0}^{10} \mathbf{F}_{[0,T]} p_1 \wedge \mathbf{F}_{\{T\}} p_2,$$

where $p_1 = ((x_{[1]} \leq 1) \wedge (x_{[2]} \leq 5))$ and $p_2 = ((x_{[1]} \leq 5) \wedge (x_{[2]} \leq 1))$. In plain English, φ states that “within 10 time units, the trajectory visits the box characterized by p_1 first and then the box corresponding to p_2 ” (see Fig. 1). We transformed this system into its MLD form (13). We formulated the constraints in (14) as a MILP. We set the cost function to maximize $\|x_0\|_\infty$ and used the Gurobi ³ MILP solver. The solution was obtained in less than 0.05 seconds on a 3GHz Dual Core MacBook Pro. We obtained $x_0 = (2.82 \ 2.82)^T$ and the following open-loop control sequence: 1 2 1 2 2 1 1 1 1 1. The corresponding trajectory is shown in Fig. 1 (a). For the same control sequence, the trajectory of the original system f with values of w drawn from a uniform distribution over $L(w^*)$ is shown in Fig. 1 (b). The resulting trajectory also satisfies the specification. The trajectory of f^* is also drawn (dashed) for comparison.

V. INFINITE HORIZON SEMANTICS

In this section, we provide the solution to Problem 2. We show that the infinite-time properties in (7) can be guaranteed using repetitive control sequences. First, we consider the case $h^\varphi = 0$ and briefly explain the results from our previous work [23]. Next we show how to extend those

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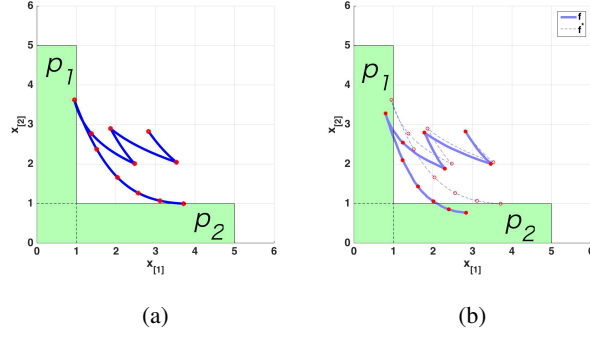


Fig. 1. Example 2: (a) The trajectory of the maximal system f^* which satisfies the specification. (b) For the same controls, the trajectory of the original system f with w drawn from an uniform distribution over $L(w^*)$.

results to global STL formulas, as stated in Problem 2. The long term behavior of the system subject to repetitive control sequences is characterized in Sec. V-B. Finally, we discuss the completeness of our results in Sec. V-C.

A. Repetitive Control Sequences

1) *s-sequences*: Consider the non-temporal specification $\varphi = (x \in \mathcal{S})$, where $\mathcal{S} \subset \mathbb{R}_+^n$ is a lower-set. We desire to keep the trajectory of the system inside \mathcal{S} for all times. These types of requirements are also known as *set-invariance* [43].

Definition 6. A *robust control invariant* (RCI) set for system (4) is a set $\Omega \subset \mathbb{R}_+^n$ such that:

$$\forall x \in \Omega, \exists u \in \mathcal{U}, f(x, u, w) \in \Omega, \forall w \in \mathcal{W}.$$

The solution to Problem 2 with non-temporal $\varphi = (x \in \mathcal{S})$ consists of finding a RCI set Ω that lies entirely in \mathcal{S} . The *maximal* RCI set inside \mathcal{S} , denoted by Ω^{\max} , provides a complete solution to the set-invariance problem. The computation of Ω^{\max} requires implementing an iterative fixed-point algorithm which is computationally intensive for MLD systems and non-convex sets (see [44], [45] for discussion). We use monotonicity to provide an alternative approach.

Definition 7. Given a lower-set \mathcal{S} and an initial condition $x_0 \in \mathbb{R}_+^n$, an *s-sequence* of length T is a control sequence $u_0^{s,x_0} u_1^{s,x_0} \dots u_{T-1}^{s,x_0}$ such that:

- 1) $x_k^{s,x_0} \in \mathcal{S}, k = 0, \dots, T$, where $x_{k+1} = f^*(x_k^{s,x_0}, u_k^{s,x_0})$, $x_0^{s,x_0} = x_0$;
- 2) $x_T^{s,x_0} \preceq x_0^{s,x_0}$.

The first condition is basically $\mathbf{G}_{[0,T]}(x \in \mathcal{S})$. The second condition $x_T \preceq x_0$ is the additional constraint that is essential for providing the following result.

Theorem 2. [23] Given an s-sequence $u_0^{s,x_0} u_1^{s,x_0} \cdots u_{T-1}^{s,x_0}$ corresponding to the initial condition x_0 , the set

$$\Omega_{\mathcal{S}} := \bigcup_{k=0}^{T-1} L(x_k^{s,x_0}), \quad (15)$$

where $x_{k+1}^{s,x_0} = f^*(x_k^{s,x_0}, u_k^{s,x_0})$, $k = 0, \dots, T-1$, $x_0^{s,x_0} = x_0$, is a RCI set inside \mathcal{S} .

Proof: For all $x \in \Omega_{\mathcal{S}}$, $\exists i \in \{0, 1, \dots, T-1\}$ such that $x \in L(x_i^{s,x_0})$. If u_i^{s,x_0} is applied, monotonicity implies $f(x, u_i^{s,x_0}, w) \preceq f^*(x_i^{s,x_0}, u_i^{s,x_0}) = x_{i+1}^{s,x_0}, \forall w \in \mathcal{W}$. It follows from (15) and the terminal condition $x_T^{s,x_0} \preceq x_0^{s,x_0}$ that $f(x, u_i^{s,x_0}, \mathcal{W}) \in L(x_{i+1}^{s,x_0})$, which ensures $f(x, u_i^{s,x_0}, w) \in \Omega_{\mathcal{S}}, \forall w \in \mathcal{W}$. ■

Example 3. Consider the system in Example 2. We wish to keep the trajectory in the set characterized by $p_1 \vee p_2$, i.e., $\mathcal{S} = L((1, 5)^T) \cup L((5, 1)^T)$. Note that this set is non-convex. We set the cost function to maximize $\|x_0\|_1$. The shortest s-sequence has $T = 5$ and is: 2 1 2 1 1. The resulting trajectory satisfying the definition of s-sequence is shown in Fig. 2 (a). The corresponding robust control invariant set is shown in Fig. 2. (b) (cyan region), which is characterized by the $x_0^{s,x_0}, x_1^{s,x_0}, \dots, x_4^{s,x_0}$ (red dots) that lie inside \mathcal{S} (green region). Note that the $[0, 2] \times [0, 2]$ portion of the coordinates in Fig. 1 is shown here for a clearer representation of the details.

2) φ -sequences: Now we consider $\mathbf{G}_{[0,\infty]}\varphi$, where φ is a bounded STL formula, and generalize the paradigm used for s-sequences.

Definition 8. Given an STL formula φ and an initial condition $x_0 \in \mathbb{R}_+^n$, a φ -sequence of length $T + h^\varphi$ is $u_0^{\varphi,x_0} u_1^{\varphi,x_0} \cdots u_{h^\varphi+T-1}^{\varphi,x_0}$ such that:

- 1) $\mathbf{x}^{\varphi,x_0}[0 : T+h^\varphi] \models \mathbf{G}_{[0,T]}\varphi$, where $x_{k+1}^{\varphi,x_0} = f^*(x_k^{\varphi,x_0}, u_k^{\varphi,x_0}), 0 \leq k \leq h^\varphi+T-1, x_0^{\varphi,x_0} = x_0$;
- 2) $\mathbf{x}^{\varphi,x_0}[T : T+h^\varphi] \preceq \mathbf{x}^{\varphi,x_0}[0 : h^\varphi]$.

Now we provide the key result of this section.

Theorem 3. Given a bounded STL formula φ and a φ -sequence $\mathbf{u}^{\varphi,x_0} := u_0^{\varphi,x_0} u_1^{\varphi,x_0} \cdots u_{T+h^\varphi-1}^{\varphi,x_0}$ corresponding to the initial condition x_0 , let μ^{ol} be the open-loop control policy corresponding to the following control sequence:

$$u_0^{\varphi,x_0} u_1^{\varphi,x_0} \cdots u_{h^\varphi-1}^{\varphi,x_0} \left(u_{h^\varphi}^{\varphi,x_0} u_{h^\varphi+1}^{\varphi,x_0} \cdots u_{h^\varphi+T-1}^{\varphi,x_0} \right)^\omega, \quad (16)$$

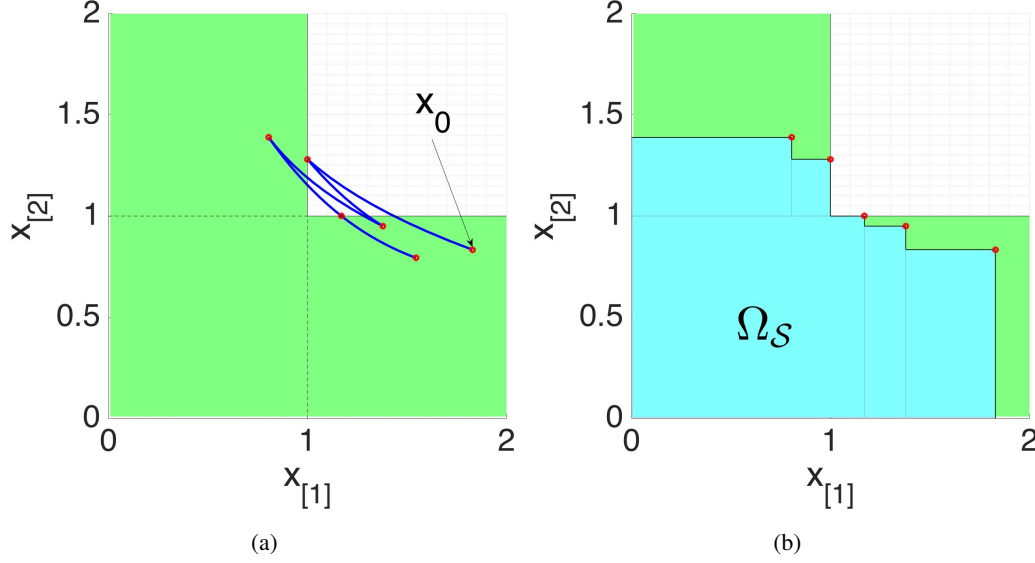


Fig. 2. Example 3: (a) The trajectory satisfying the conditions of s-sequences. (b) The corresponding robust control invariant set inside S .

Then $\mathbf{x}(x'_0, \mu^{ol}, \mathbf{w}) \models \mathbf{G}_{[0,\infty]}\varphi, \forall \mathbf{w} \in \mathcal{W}^\omega, \forall x'_0 \in L(x_0)$.

Proof: First, we prove that $\mathbf{x}(x'_0, \mu^{ol}, \mathbf{w}^*) \models \mathbf{G}_{[0,\infty]}\varphi$, where $\mathbf{w}^* = (w^*)^\omega$. Note that for all $k \geq T$, we have $\mu_k^{ol}(x_0^\varphi) = u_{h^\varphi + \text{rem}(k-h^\varphi, T)}^{\varphi, x_0}$.

Foremost, we show that $x_{k+T+h^\varphi}^{\varphi, x_0} \preceq x_{k+h^\varphi}^{\varphi, x_0}, \forall k \in \mathbb{N}$. We prove by induction. The basis ($k = 0$) $x_{T+h^\varphi}^{\varphi, x_0} \preceq x_{h^\varphi}^{\varphi, x_0}$ is given by the terminal condition in Definition 8. The inductive step $x_{k+T+h^\varphi}^{\varphi, x_0} \preceq x_{k+h^\varphi}^{\varphi, x_0}$ implying $x_{k+T+h^\varphi+1}^{\varphi, x_0} \preceq x_{k+h^\varphi+1}^{\varphi, x_0}$ is verified by monotonicity:

$$\begin{aligned}
 x_{k+T+h^\varphi+1}^{\varphi, x_0} &= f^*(x_{k+T+h^\varphi}^{\varphi, x_0}, u_{h^\varphi + \text{rem}(k+T-h^\varphi, T)}^{\varphi, x_0}) \\
 &= f^*(x_{k+T+h^\varphi}^{\varphi, x_0}, u_{h^\varphi + \text{rem}(k-h^\varphi, T)}^{\varphi, x_0}) \\
 &\preceq f^*(x_{k+h^\varphi}^{\varphi, x_0}, u_{h^\varphi + \text{rem}(k-h^\varphi, T)}^{\varphi, x_0}) \\
 &= x_{k+h^\varphi+1}^{\varphi, x_0}.
 \end{aligned}$$

By adding the constraints of $\mathbf{x}^{\varphi, x_0}[T : T + h^\varphi] \preceq \mathbf{x}^{\varphi, x_0}[0 : h^\varphi]$, it follows that $x_{k+T}^{\varphi, x_0} \preceq x_k^{\varphi, x_0}, \forall k \in \mathbb{N}$. Consequently, $\mathbf{x}^{\varphi, x_0}[k+T : k+T+h^\varphi] \preceq \mathbf{x}^{\varphi, x_0}[k : k+h^\varphi], \forall k \in \mathbb{N}$. Similarly, $\mathbf{x}^{\varphi, x_0}[k+T : k+T+h^\varphi] \preceq \mathbf{x}^{\varphi, x_0}[\text{rem}(k, T) : \text{rem}(k, T) + h^\varphi], \forall k \in \mathbb{N}$. It follows from the first condition in Definition 8 that $\mathbf{x}^{\varphi, x_0}[i : i+h^\varphi] \models \varphi, i = 0, 1, \dots, T$. Therefore, by the virtue of Lemma 1, $\mathbf{x}^{\varphi, x_0}[k : k+h^\varphi] \models \varphi, \forall k \in \mathbb{N}$, which immediately implies $\mathbf{x}^{\varphi, x_0}[0] \models \mathbf{G}_{[0,\infty]}\varphi$.

Now we consider the original system (4) $x'_{k+1} = f(x'_k, \mu_k^{ol}(x_0), w_k), x'_0 \in L(x_0)$. From monotonicity and Assumption 1, we have $x'_k \preceq x_k^\varphi, \forall k \in \mathbb{N}$. Thus $\mathbf{x}(x_0, \mu^{ol}, \mathbf{w})[0] \preceq \mathbf{x}^{\varphi, x_0}[0], \forall x_0 \in$

$L(x_0^\varphi), \forall w \in \mathcal{W}^\omega$. Following Lemma 1, we conclude $\mathbf{x}(x_0', \mu^{ol}, \mathbf{w})[0] \models \mathbf{G}_{[0,\infty]}\varphi, \forall x_0' \in L(x_0), \forall \mathbf{w} \in \mathcal{W}^\omega$, and the proof is complete. ■

Note that s-sequences are special cases of φ -sequences. A φ -sequence consists of a initialization segment of length h^φ and a repetitive segment of length T . In case $h^\varphi = 0$, an φ -sequence becomes a s-sequence which only consists of the repetitive segment.

The computation of a φ -sequence requires solving an MILP for $\mathbf{G}_{[0,T]}\varphi$ (similar to Problem 1) with an additional set of constraints in $\mathbf{x}^{\varphi,x_0}[T : T+h^\varphi] \preceq \mathbf{x}^{\varphi,x_0}[0 : h^\varphi]$. We are usually interested in the shortest φ -sequence since its computation requires the smallest MILP. Algorithmically, we start from $T = 1$ and implement $T \leftarrow T + 1$ until the MILP formulating the conditions in Definition 8 becomes feasible and a φ -sequence is found. In Sec. V-C, we discuss the existence of finite length φ -sequences.

We now explain the analogous version of RCI sets in Theorem 2 for STL formulas. We introduce some additional notation.

Definition 9. Given a bounded STL formula φ over predicates in the form (6), the *language realization set* (LRS) is defined as [46]:

$$\mathcal{L}^\varphi := \{ \mathbf{x}[\cdot : \cdot + h^\varphi] \in \mathcal{X}^{h^\varphi} \mid \mathbf{x}[\cdot : \cdot + h^\varphi] \models \varphi \}. \quad (17)$$

Proposition 2. The set \mathcal{L}^φ is a lower-set.

Proof: For all $\mathbf{x}[\cdot : \cdot + h^\varphi] \in \mathcal{L}^\varphi$ and for all $\mathbf{x}'[\cdot : \cdot + h^\varphi] \preceq \mathbf{x}[\cdot : \cdot + h^\varphi]$, we have from Lemma 1 that $\mathbf{x}'[\cdot : \cdot + h^\varphi] \models \varphi$ hence $\mathbf{x}'[\cdot : \cdot + h^\varphi] \in LRS(\varphi)$, which indicates $LRS(\varphi)$ is a lower set. ■

The language realization set is analogous to the safe set when $h^\varphi = 0$ and it can be interpreted as the safe set in h^φ -length trajectory space.

Proposition 3. If there exists a φ -sequence $u_0^{\varphi,x_0} u_1^{\varphi,x_0} \cdots u_{h^\varphi+T-1}^{\varphi,x_0}$ corresponding to the initial condition x_0 , then the following set:

$$\Omega_{\mathcal{L}^\varphi} := \bigcup_{k=0}^{T-1} L(\mathbf{x}^{\varphi,x_0}[k : k + h^\varphi]), \quad (18)$$

where $x_{k+1}^{\varphi,x_0} = f^*(x_k^{\varphi,x_0}, u_k^{\varphi,x_0}), k = 0, 1, \dots, h^\varphi + T - 1$, is a RCI set in \mathcal{L}^φ in the sense that

$$\begin{aligned} \forall \quad x'_0 x'_1 \cdots x'_{h^\varphi} \in \Omega_{\mathcal{L}^\varphi}, \exists u \in \mathcal{U}, \\ x'_1 x'_2 \cdots x'_{h^\varphi} f(x'_{h^\varphi}, u, w) \in \Omega_{\mathcal{L}^\varphi}, \forall w \in \mathcal{W}. \end{aligned} \quad (19)$$

Proof: The procedure is analogous to the proof of Theorem 2. For any $x'_0 x'_1 \cdots x'_{h^\varphi} \in \Omega^\varphi$, there exists $i \in \{0, 1, \dots, T-1\}$ such that $x'_0 x'_1 \cdots x'_{h^\varphi} \in L(\mathbf{x}^{\varphi, x_0}[i : i + h^\varphi])$, which is equivalent to $x'_k \preceq x_{i+k}^{\varphi, x_0}, k = 0, 1, \dots, h^\varphi$. On one hand, we have $x_{i+1}^{\varphi, x_0} \cdots x_{i+h^\varphi}^{\varphi, x_0} f^*(x_{i+h^\varphi}^{\varphi, x_0}, u_{i+h^\varphi}^{\varphi, x_0}) \in \Omega_{\mathcal{L}^\varphi}$. On the other hand, we have $x'_1 \preceq x_{i+1}^{\varphi, x_0}, \dots, x'_{h^\varphi} \preceq x_{i+h^\varphi}^{\varphi, x_0}$, and by applying $u_{i+h^\varphi}^{\varphi, x_0}$, monotonicity implies

$$\begin{aligned} f(x'_{h^\varphi}, u_{i+h^\varphi}^{\varphi, x_0}, w) &\preceq f^*(x_{i+h^\varphi}^{\varphi, x_0}, u_{i+h^\varphi}^{\varphi, x_0}), \forall w \in \mathcal{W} \\ \Rightarrow x'_1 x'_2 \cdots x'_{h^\varphi} f(x'_{h^\varphi}, u_{i+h^\varphi}^{\varphi, x_0}, w) &\in \\ L(x_{i+1}^{\varphi, x_0} x_{i+2}^{\varphi, x_0} \cdots x_{i+h^\varphi}^{\varphi, x_0} f^*(x_{i+h^\varphi}^{\varphi, x_0}, u_{i+h^\varphi}^{\varphi, x_0})) &, \forall w \in \mathcal{W}. \end{aligned}$$

Since $x_{i+1}^{\varphi, x_0} x_{i+2}^{\varphi, x_0} \cdots x_{i+h^\varphi}^{\varphi, x_0} f^*(x_{i+h^\varphi}^{\varphi, x_0}, u_{i+h^\varphi}^{\varphi, x_0}) \in \Omega_{\mathcal{L}^\varphi}$, we also have $x'_1 x'_2 \cdots x'_{h^\varphi} f(x'_{h^\varphi}, u_{i+h^\varphi}^{\varphi, x_0}, w) \in \Omega_{\mathcal{L}^\varphi}, \forall w \in \mathcal{W}$ and the proof is complete. ■

It is worth to mention that the definition of the language realization set is independent of the system constraints. For some $\mathbf{x}[0 : h^\varphi] \in \Omega_{\mathcal{L}^\varphi}$, $x_0 x_1 \cdots x_h^\varphi \cdots$ may not be consistent with system (4). Therefore, a φ -sequence consists of a transient segment to reach a point in $\Omega_{\mathcal{L}^\varphi}$ which is consistent with the system constraints. Afterwards, the repetitive segment ensures remaining in $\Omega_{\mathcal{L}^\varphi}$ for all times.

B. Limit Sets and Attractive Region

Here we characterize the long-term behavior of the trajectories if the control sequence (16) is applied. We both study the behavior of the maximal system f^* and the original system f . For a run $\mathbf{x} = x_0 x_1 \cdots$, we define the ω -limit set as

$$\omega(\mathbf{x}) = \bigcap_{\tau=0}^{\infty} cl(\{x_t | t \geq \tau\}), \quad (20)$$

where cl stands for set-closure. The following lemma is a basic result in real analysis, which is also known as the *monotone convergence theorem* [47]:

Lemma 2. If a real-valued sequence is monotonically decreasing (increasing) and it is bounded from below (above), then it converges to its infimum (supremum).

Theorem 4. Given a φ -sequence corresponding to the initial condition x_0 , the following results hold:

- 1) the ω -limit set of run generated by $x_{k+1}^{\varphi, x_0} = f^*(x_k^{\varphi, x_0}, u_k)$, $x_0^{\varphi, x_0} = x_0$, where values for u_k are given as (16), is non-empty and corresponds to the following periodical orbit:

$$(x_\infty^{\varphi, \infty}, x_1^{\varphi, \infty}, \dots, x_{T-1}^{\varphi, \infty})^\omega, \quad (21)$$

where $x_k^{\varphi, \infty} := \lim_{c \rightarrow \infty} x_{k+cT}^{\varphi, x_0}$, $k = 0, \dots, T-1$.

- 2) the following set is an attractor for all trajectories of the original system $x_{t+1} = f(x_t, u_t, w_t)$, values for u_t given as (16), that originate from $L(x_0^\varphi)$

$$\Gamma_\varphi := \bigcup_{k=0}^{T-1} L(x_k^{\varphi, \infty}). \quad (22)$$

Proof: 1) Continue from proof of Theorem 3: Since $x_{k+T}^{\varphi, x_0} \preceq x_k^{\varphi, x_0}$, $\forall k \in \mathbb{N}$, we obtain that $x_{cT+k}^{\varphi, x_0} \preceq x_k^{\varphi, x_0}$, $\forall c \in \mathbb{N}, \forall k \in \mathbb{N}$. Therefore each vector component of the real sequence $x_k^{\varphi, x_0} x_{k+T}^{\varphi, x_0} x_{k+2T}^{\varphi, x_0} \dots$, $k = 0, 1, \dots, T-1$ is monotonically decreasing. Since the values are bounded from below (zero), then by virtue of Lemma 2, the sequence converges hence $\lim_{c \rightarrow \infty} x_{k+cT}^{\varphi, x_0}$ exists, $k = 0, 1, \dots, T-1$. 2) For all runs of the original system $x_0 x_1 x_2 \dots$, monotonicity implies that $x_t \preceq x_t^{\varphi, x_0}$, $\forall t \in \mathbb{N}$. Therefore $x_t \in L(x_t^{\varphi, \infty} \bmod T)$ becomes true as $t \rightarrow \infty$. ■

Example 4. Consider the system in Example 2. We wish to satisfy $\mathbf{G}_{[0, \infty]} \varphi$, where

$$\varphi = \mathbf{F}_{[0, 7]} p_1 \wedge \mathbf{F}_{[0, 7]} p_2.$$

This specification requires that regions p_1 and p_2 are persistently visited while the maximum time between two subsequent visits is not greater than 7. We find a φ -sequence solving a MILP for $T = 8$ while maximizing $\|x_0\|_1$. The control sequence is $= 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ (1 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1)^\omega$. The open-loop trajectory of the maximal system f^* satisfying the definition of φ -sequence is shown in Fig. 4 (a). We also find that the ω -limit set of f^* is a 8-periodic orbit (shown in Fig. 4 (b)). The attracting set Γ^φ is characterized by $x^{\infty, k}$, $k = 0, 1, \dots, 7$, which are shown using red dots in Fig. 4 (c). A trajectory of the original system f starting from x_0 with values of w chosen uniformly from \mathcal{W} is shown in Fig. 4 (d). It is observed that the trajectory satisfies the specification while eventually reaching Γ_φ and staying there afterwards.

C. Existence of φ -sequences

In this section, we discuss the necessity of existence of φ -sequences. First, we introduce some additional notation and terminology. For a given system f with disturbance set \mathcal{W} , we denote the maximal disturbance by $M(\mathcal{W})$, which was denoted by w^* earlier. We characterize the necessity of φ -sequences with respect to perturbations in the disturbance set \mathcal{W} .

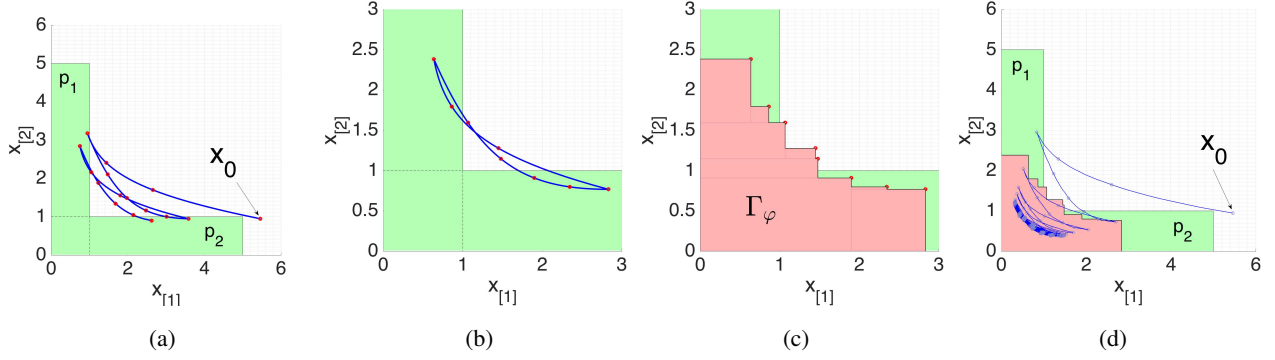


Fig. 3. Example 4: (a) The trajectory of the maximal system f^* for length $T + h^\varphi = 8 + 7 = 15$ which satisfies the definition of φ -sequence. (b) The ω -limit set (red dots) of f^* using the open-loop control sequence, which is a 8-periodic orbit. (c) The attracting region (red) characterized by $x^{\infty,k}$, $k = 0, 1, \dots, 8$, which are shown using red dots. (d) A trajectory of the original system using controls (16) while disturbances are drawn uniformly from $L(w^*)$.

Definition 10. System (4) is *strongly monotone with respect to the maximal disturbance* (SMMD) if for all $\epsilon > 0$, there exists a disturbance set \mathcal{W}' such that

$$f(x, u, M(\mathcal{W})) + 1_n \epsilon \preceq f(x, u, M(\mathcal{W}')), \forall x \in \mathcal{X}, \forall u \in \mathcal{U}. \quad (23)$$

Theorem 5. Suppose the system (4) is SMMD and the set \mathcal{L}^φ is bounded. Given $\epsilon > 0$, the disturbance set is altered to \mathcal{W}' such that $f(x, u, M(\mathcal{W})) + 1_n \epsilon \preceq f(x, u, M(\mathcal{W}')), \forall x \in \mathcal{X}, \forall u \in \mathcal{U}$. If there exists a control policy μ and an initial condition x_0 such that $\mathbf{x}(x_0, \mu, \mathbf{w}') \models \mathbf{G}_{[0,\infty]}\varphi, \forall \mathbf{w}' \in \mathcal{W}'^{\omega}$, then there exists at least one φ -sequence of length $T + h^\varphi$ for the original system such that:

$$T \leq \frac{A}{\epsilon^{n(h^\varphi+1)}}, \quad (24)$$

where A is a constant depending on \mathcal{L}^φ .

Proof: We define the following terms that are only used for this proof. A real valued sequence $r_0 r_1 \dots, r_k \in \mathbb{R}^n, k \in \mathbb{N}$, is *forwardly ordered* if there exists $n_T, n_0 \in \mathbb{N}, n_T \geq n_0$, such that $r_{n_T} \preceq r_{n_0}$. Given a bounded set $\mathcal{C} \subset \mathbb{R}^{n(h^\varphi+1)}$, we define the diameter $d(\mathcal{C})$ as $\inf d$ such that $s_1 \preceq s_2 + 1d, \forall s_1, s_2 \in \mathcal{C}$ (e.g., the diameter of an axis-aligned hyper-box is equal to the length of its largest side). Consider a partition of \mathcal{L}^φ by a finite number of cells, where the diameter of each cell is less than ϵ . The maximum number of cells required for such a partition

is proportional to $\frac{1}{\epsilon^{n(h^\varphi+1)}}$. We have $N \leq \frac{A}{\epsilon^{n(h^\varphi+1)}}$, where A is a constant dependent on the shape and volume of \mathcal{L}^φ . A conservative upper bound on A can be given as follows. We define:

$$a = \underset{i=1, \dots, n(h^\varphi+1), \mathbf{x}[:, : + h^\varphi] \in \mathcal{L}^\varphi}{\operatorname{argmin}} \mathbf{x}[:, : + h^\varphi] \preceq \underbrace{1_n a \cdots 1_n a}_{h^\varphi+1 \text{ times}}.$$

Since \mathcal{L}^φ is bounded and closed in $\mathbb{R}_+^{n(h^\varphi+1)}$, a exists. We have $\mathcal{L}^\varphi \subseteq L(1_n(h^\varphi+1)a)$. Let A be $a^{n(h^\varphi+1)}$ (volume of $L(1_n(h^\varphi+1)a)$). The hyper-box $L(1_n(h^\varphi+1)a)$ is partitioned into $\frac{A}{\epsilon^{n(h^\varphi+1)}}$ number of equally sized cubic cells with side length of ϵ . Such a grid partitions \mathcal{L}^φ to at most $\frac{A}{\epsilon^{n(h^\varphi+1)}}$ number of cells where the diameter of each cell is not greater than ϵ .

Since there exist μ such that $\mathbf{x}(x_0, \mu, \mathbf{w}') \models \mathbf{G}_{[0, \infty]} \varphi, \forall \mathbf{w}' \in \mathcal{W}'^\omega$, there exist at least one run satisfying φ for system $x'_{k+1} = f(x'_k, u'_k, M(\mathcal{W}'))$. Let $x'_0, x'_1, \dots, x'_{N+h^\varphi}$ be the first $N + h^\varphi + 1$ time points of a such a run. Note that $\mathbf{x}'[k : k + h^\varphi] \in \mathcal{L}^\varphi, 0 \leq k \leq N$.

case 1: If the sequence $\mathbf{x}'[0 : h^\varphi] \mathbf{x}'[1 : 1 + h^\varphi] \cdots \mathbf{x}'[N : N + h^\varphi]$ is forwardly ordered, then there exists $0 \leq k_1 \leq k_2 \leq N$ such that $\mathbf{x}'[k_1 : k_1 + h^\varphi] \preceq \mathbf{x}'[k_2 : k_2 + h^\varphi]$. Therefore, by reconsidering the start of the sequence from k_1 , the conditions in the Definition of φ -sequences is met for the system with the adversarial set \mathcal{W}' . Thus, a φ -sequence of length $T + h^\varphi$ exists such that $T = k_2 - k_1, T \leq N$. Monotonicity and (23) imply the same φ -sequence is also valid for the system with the adversarial set \mathcal{W} .

case 2: Suppose the sequence $\mathbf{x}'[0 : h^\varphi] \mathbf{x}'[1 : 1 + h^\varphi] \cdots \mathbf{x}'[N : N + h^\varphi]$ is not forwardly ordered. Consider a partition of \mathcal{L}^φ with cells diameter less than ϵ . By the virtue of *pigeonhole principle*, there exists a cell such that two contains at least two time points $\mathbf{x}'[k_1 : k_1 + h^\varphi]$ and $\mathbf{x}'[k_2 : k_2 + h^\varphi], 0 \leq k_1 \leq k_2 \leq N$. From the assumption on the diameter of the cells we have $\mathbf{x}'[k_2 : k_2 + h^\varphi] \preceq \mathbf{x}'[k_1 : k_1 + h^\varphi] + 1_{n(h^\varphi+1)}\epsilon$. Let $T = k_2 - k_1$. Now consider system $x_{k+1} = f(x_k, u_k, M(\mathcal{W}))$ with initial condition $x_0 = x'_{k_1}$, and the open-loop control sequence $u_k = u'_{k+k_1}, \forall k \in 0, 1, \dots, T-1$. It follows from (23) that $\mathbf{x}[T : T + h^\varphi] + 1_{n(h^\varphi+1)}\epsilon \preceq \mathbf{x}'[k_2 : k_2 + h^\varphi]$, which by substituting in $\mathbf{x}'[k_2 : k_2 + h^\varphi] \preceq \mathbf{x}'[k_1 : k_1 + h^\varphi] + 1_{n(h^\varphi+1)}\epsilon$ yields $\mathbf{x}[T : T + h^\varphi] \preceq \mathbf{x}'[k_1 : k_1 + h^\varphi]$, or equivalently, $\mathbf{x}[T : T + h^\varphi] \preceq \mathbf{x}[0 : h^\varphi]$. Therefore, the sequence $u'_{k_1-h^\varphi} u'_{k_1-h^\varphi+1} \cdots u'_{k_2-k_1}$ becomes a φ -sequence. Since $T \leq N$, we have $T \leq \frac{A}{\epsilon^{n(h^\varphi+1)}}$ and the proof is complete. \blacksquare

Theorem 5 states that if we can not find a φ -sequence, then it is very likely that a correct control policy for Problem 2 does not exist. If it exists, it is *fragile* in the sense that a slight increase in the effect of the disturbances makes the policy invalid. The relation between the

fragility and the length of the φ -sequence suggests that by performing the search for longer φ -sequences (which are computationally more difficult), the bound for fragility becomes smaller, implying that a correct control policy (if exists) is close to the limits (i.e., robustness score is close to zero, or the constraints are barely satisfied). In practice, the bounds in Theorem 5 are very conservative and one may desire to find tighter bounds for specific applications.

Example 5. Consider Example 3. Suppose that there does not exist an s-sequence of length smaller than 144 with maximal disturbance w^* . The volume (area in this 2D case) of region corresponding to $p_1 \vee p_2$ is 9. Therefore, \mathcal{S} can be partitioned into 144 equally sized square cells with side length 0.25. Note that we have $\epsilon^2 \geq 9/T$. Since the disturbance is additive, it follows that a control strategy μ such that $\mathbf{x}(x_0, \mu, \mathbf{w}') \models \mathbf{G}_{[0,\infty]}(x \in \mathcal{S}), \forall \mathbf{w}' \in \mathcal{W}'^\omega$, does not exist for any $x_0 \in \mathbb{R}_+^n$ if $\mathcal{W}' = L(w^* + (0.25, 0.25)^T)$.

VI. MODEL PREDICTIVE CONTROL

In this section, we provide a solution to Problem 3. We assume full knowledge of the history of state. As mentioned in Sec. III, u_t is given by the solution to the optimization problem (8). Furthermore, the cost function J is also assumed to be non-decreasing with respect to the state values hence the system constraints are replaced with those of the maximal system. First, we explain the MPC setup for global STL formulas. Next, we prove that the proposed framework is guaranteed to generate runs that satisfy the global STL specification (7).

Given planning horizon H , the states that are predictable at time t using controls in u_t^H are $x_{1|t}, x_{2|t}, \dots, x_{H|t}$. The requirement $\mathbf{x}[0] \models \mathbf{G}_{[0,\infty]}\varphi$ can be interpreted as $\mathbf{x}[t : t + h^\varphi] \models \varphi, \forall t \in \mathbb{N}$, or equivalently, $\mathbf{x}[t : t + h^\varphi] \in \mathcal{L}^\varphi, \forall t \in \mathbb{N}$. Note that x_{t+1} is the last value appearing in $\mathbf{x}[t - h^\varphi + 1 : t + 1]$ (for now, we assume that $t \geq h^\varphi$. The case of $t < h^\varphi$ is explained later.) and x_{t+H} is the last value appearing in $\mathbf{x}[t + H - h^\varphi : t + H]$. Therefore, using the predictions for $x_{t+1}, x_{t+2}, \dots, x_{t+H}$, which are $x_{1|t}, x_{2|t}, \dots, x_{H|t}$, we need to enforce the constraint $\mathbf{x}[t - h^\varphi + 1, t + H] \models \mathbf{G}_{[t-h^\varphi+1, t+H-h^\varphi]}\varphi$. Note that the values in $\mathbf{x}[\tau : \tau + h^\varphi]$ are independent of the values in x_t^H for $\tau \leq t - h^\varphi$ and are not fully available for $\tau > t + H - h^\varphi$. Thus, $[t - h^\varphi + 1, t + H - h^\varphi]$ is the time window for imposing constraints at time t [14].

The MPC optimization problem is initially written as (we do not solve it as explained shortly):

$$\begin{aligned} & \text{minimize} && J(x_t^H, u_t^H), \\ & \text{s.t.} && x_{k+1|t} = f^*(x_{k|t}, u_{k|t}), k = 0, \dots, H-1, \\ & && \mathbf{x}[t - h^\varphi + 1, t + H] \models \mathbf{G}_{[t-h^\varphi+1, t+H-h^\varphi]}\varphi, \end{aligned} \tag{25}$$

where $x_t^H = x_{1|t}, \dots, x_{H-1|t}$, and $\mathbf{x}[t+1, t+H] = x_{1|t} \cdots x_{H|t}$ (which are predictions, not actual states). The set of constraints in (25) requires the knowledge of $x_{t-h^\varphi+1}x_{t-h^\varphi+2} \cdots x_t$. Thus, the proposed control policy requires a finite memory for the history of last h^φ states.

Proposition 4. If the optimization problem (25) is feasible for all times, then $\mathbf{x}[0] \models \mathbf{G}_{[0,\infty]}\varphi$.

Proof. We prove that feasibility implies $\mathbf{x}[t - h^\varphi : t] \models \varphi, \forall t \geq h^\varphi$, using induction over t . The case for $t = h^\varphi$ is assumed to be true (see the end of this section). At time t , we apply $u_t = u_{0|t}$. Thus, $x_{t+1} = f(x_t, u_t, w_t)$. We have $x_{t+1} \preceq x_{1|t} = f^*(x_t, u_t)$ and $\mathbf{x}[t - h^\varphi + 1, t + 1] = x_{t-h^\varphi+1} \cdots x_{t+1}$. From feasibility at time t we already had $x_{t-h^\varphi+1} \cdots x_t x_{1|t} \models \varphi$. Since $x_{t-h^\varphi+1} \cdots x_t x_{t+1} \preceq x_{t-h^\varphi+1} \cdots x_t x_{1|t}$, Lemma 1 implies $\mathbf{x}[t - h^\varphi + 1, t + 1] \models \varphi$. \square

However, persistent feasibility of the MPC setup in (25) is not guaranteed. We address this issue for the remainder of this section.

Definition 11. An MPC strategy is *recursively feasible* if, for all $t \in \mathbb{N}$, the control at time t is selected such that the MPC optimization problem at $t + 1$ becomes feasible.

Our goal is to modify (25) such that it becomes recursively feasible. It is known that adding a (the maximal) RCI set acting as a terminal constraint is sufficient (and necessary) to guarantee recursive feasibility [48]. We follow the set-invariance approach by using the results from the previous section to establish recursive feasibility. We add the terminal constraint $\mathbf{x}[t + H - h^\varphi : t] \in \Omega_{\mathcal{L}^\varphi}$ to (25) to obtain:

$$\begin{aligned} u_t^H &= \underset{u_t^H \in \mathcal{U}^H}{\operatorname{argmin}} J(x_t^H, u_t^H), \\ \text{s.t. } \quad &x_{k+1|t} = f^*(x_{k|t}, u_{k|t}), k = 0, \dots, H-1, \\ &\mathbf{x}[t - h^\varphi + 1, t + H] \models \mathbf{G}_{[t-h^\varphi+1, t+H-h^\varphi]}\varphi, \\ &\mathbf{x}[t + H - h^\varphi : t + H] \in \Omega_{\mathcal{L}^\varphi}. \end{aligned} \tag{26}$$

Proposition 5. The MPC optimization problem (26) is recursively feasible.

Proof: Suppose $u_t^H = u_{0|t}, u_{1|t}, \dots, u_{H-1|t}$ and $x_t^H = x_{t+1|t}, x_{t+2|t}, \dots, x_{t+H-1|t}$ is a feasible solution for (26) at time t . Since $\Omega_{\mathcal{L}^\varphi}$ is a RCI set, there exist $u^r \in \mathcal{U}$ such that $\mathbf{x}[t + H + 1 - h^\varphi : t + H + 1] = x_{H-h^\varphi+1|t}x_{H-h^\varphi+2|t} \cdots x_{H|t}f(x_{H|t}, u^r, w) \in \Omega_{\mathcal{L}^\varphi}, \forall w \in \mathcal{W}$. Suppose $u_{0|t}$ applied to the system. From monotonicity we have $x_{t+1} = f(x_t, u_{0|t}, w) \preceq f^*(x_t, u_{0|t}) = x_{1|t}, \forall w \in \mathcal{W}$. We now show that the optimization problem at time $t + 1$ is feasible because at least one u_{t+1}^H exists that satisfies all the constraints. This is $u_{t+1}^H = u_{1|t}, u_{2|t}, \dots, u_{H|t}, u^r$. We have

$x_{t+1}^H = x_{1|t+1}, \dots, x_{H|t+1}$. We already showed that $x_{t+1} = x_{0|t+1} \preceq x_{1|t}$. By induction and using monotonicity, it follows that $x_{k|t+1} \preceq x_{k+1|t}, k = 1, \dots, H-2$. Therefore, we have $x_{t-h^\varphi+2} \dots x_{t+1} x_{1|t+1} \dots x_{H-1|t+1} \preceq x_{t-h^\varphi+2} \dots x_{1|t} x_{2|t} \dots x_{H|t}$, which establishes (from Lemma 1) $x_{t-h^\varphi+2} \dots x_{t+1} x_{1|t+1} \dots x_{H-1|t+1} \models \mathbf{G}_{[t-h^\varphi+2, t+H-h^\varphi]}$. It remains to show that $\mathbf{x}[t+H+1-h^\varphi : t+H+1] = x_{H+1-h^\varphi|t} \dots x_{H|t+1} \models \varphi$. This follows from invariance. Note that $x_{H|t+1} = f^*(x_{H|t}, u^r)$. Therefore $x_{H+1-h^\varphi|t} \dots x_{H|t+1} \in \Omega_{\mathcal{L}^\varphi}$, and since $\Omega_{\mathcal{L}^\varphi} \in \mathcal{L}^\varphi$, we have $x_{H+1-h^\varphi|t} \dots x_{H|t+1} \models \varphi$. Therefore, by taking $u_{t+1}^H = u_{1|t}, u_{2|t}, \dots, u_{H|t}, u^r$, all constraints are satisfied, hence establishing recursive feasibility. ■

The MPC optimization problem is also converted into a MILP problem. It is computationally easier to solve the optimization problem in (26) by solving T MILPs:

$$\begin{aligned} u_t^H &= \underset{u_t^H \in \mathcal{U}^H, i=0, \dots, T-1}{\operatorname{argmin}} J(x_t^H, u_t^H), \\ \text{s.t.} \quad & x_{k+1|t} = f^*(x_{k|t}, u_{k|t}), k = 0, \dots, H-1, \\ & \mathbf{x}[t-h^\varphi+1, t+H] \models \mathbf{G}_{[t-h^\varphi+1, t+H-h^\varphi]}\varphi, \\ & \mathbf{x}[t+H-h^\varphi : t+H] \in L(\mathbf{x}^{\varphi, x_0}[i : i+h^\varphi]). \end{aligned} \tag{27}$$

Note that all MILPs can be aggregated into a single large MILP in the expense of additional constraints for capturing non-convexities of the terminal condition.

Finally, consider $t < h^\varphi$. In this case, we require $H \geq h^\varphi$ and replace the interval $[t-h^\varphi+1, t+H-h^\varphi]$ with $[0, t+H-h^\varphi]$ for $t < h^\varphi$ in (27). For applications where initialization is not important in long-term (like traffic management), a simpler approach is to initialize the MPC from $t = h^\varphi$ and assume all previous state values are zero (hence all the past predicates are evaluated as true).

Remark 3. In our previous work on STL MPC of linear systems [14], we did not establish recursive feasibility hence in order to recover from possible infeasibility issues we maximized the STL robustness score (a negative value) whenever the MPC optimization problem became infeasible. Large un-modeled disturbances and initial conditions outside \mathcal{X}_0^{\max} may lead to infeasibility. Thus, we can use the formalism in [14] to recover from infeasibility, whenever encountered, with minimal violation of the specification.

VII. APPLICATION TO TRAFFIC MANAGEMENT

In this section, we explain how to apply our methods to traffic management. First, the model that we use for traffic networks is explained, which is similar to the one in [49] but freeways

are also modeled. Next, the monotonicity properties of the model are discussed. We show that there exists a congestion-free set in the state-space in which the traffic dynamics is monotone. Finally, a case study on a mixed urban and freeway network is presented.

A. Model

The topology of the network is described by a directed graph $(\mathcal{V}, \mathcal{L})$, where \mathcal{V} is the set of nodes and \mathcal{L} is the set of edges. Each $l \in \mathcal{L}$ represents a one-way traffic link from tail node $\tau(l) \in \mathcal{V} \cup \emptyset$ to head node $\eta(l) \in \mathcal{V}$, where $\tau(l) = \emptyset$ stands for links originating from outside of the network. We distinguish between three types of links based on their control actuations: 1) \mathcal{L}_r : road links actuated by traffic lights, 2) \mathcal{L}_o : freeway on-ramps actuated by ramp meters, 3) \mathcal{L}_f : freeway segments which are not directly controlled. Freeway off-ramps are treated the same way as the roads. Uncontrolled roads are also treated the same as freeways. We have $\mathcal{L}_r \cup \mathcal{L}_o \cup \mathcal{L}_f = \mathcal{L}$.

Remark 4. Some works, e.g. [17], consider control over freeway links by varying speed limits, which adds to the control power but requires the existence of such a control architecture within the infrastructure. We do not consider this type of control actuation in this paper but it can easily be incorporated into our model by modeling freeways links the same way as on-ramps, where the speed limit becomes analogous to the ramp meter input.

The number of vehicles on link l at time t is represented by $x_{[l],t} \in [0, c_l]$, which is assumed to be a continuous variable, and c_l is the capacity of l . In other words, vehicular movements are treated as fluid-like flow in our model. The number of vehicles that are able to flow out of l in one time step, if link l is actuated, is:

$$q_{[l],t} := \min \left\{ x_{[l],t}, \bar{q}_l, \min_{\{l' : \tau(l') = \eta(l)\}} \frac{\alpha_{l:l'}}{\beta_{l:l'}} (c_{l'} - x_{[l'],t}) \right\}, \quad (28)$$

where \bar{q}_l is the maximum outflow of link l in one time step, which is physically related to the speed of the vehicles. The last argument in the minimizer determines the minimum supply available in the downstream links of l , where $\alpha_{l:l'} \in [0, 1]$ is the capacity ratio of link l' available to vehicles arriving from link l (typically portion of the lanes), $\beta_{l:l'} \in [0, 1]$ is the ratio of the vehicles in l that flow into l' (turning ratio). For simplicity, we assume capacity ratios and turning ratios are constants. System state is represented by $x \in \mathbb{R}_+^n : \{x_{[l]}\}_{l \in \mathcal{L}}$, where n is the number of the links in the network. The state space is $\mathcal{X} := \prod_{l \in \mathcal{L}} [0, c_l]$.

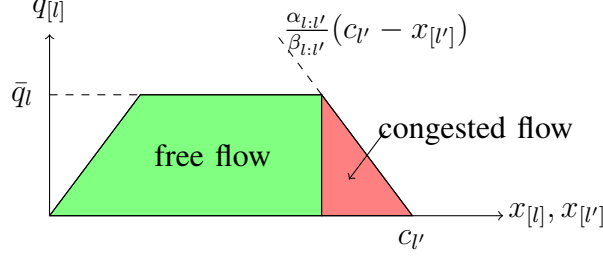


Fig. 4. The fundamental diagram. The flow out of link l drops if the number of vehicles on the immediate downstream link l' is close to its capacity. The congestion is defined by this blocking behavior.

A schematic diagram illustrating the behavior of $q[l]$ with respect to the state variables $x[l], x[l']$ - which is known as the *fundamental diagram* in the traffic literature [50] - is shown in Fig. 4. It is observed that the traffic flow from a particular link drops if one (or more) of its downstream links do not have enough capacity to accommodate the incoming flow. In this case (when the last argument in (28) is the minimizer), we say the traffic flow is *congested*. Otherwise, the traffic flow is *free*. This motivates the following definition:

Definition 12. The *congestion-free set*, denoted by Π , is defined as the following region in the state space:

$$\Pi := \left\{ x \in \mathcal{X} \mid \min\{x_l, \bar{q}_l\} \leq \frac{\alpha_{l:l'}}{\beta_{l:l'}}(c_{l'} - x_{l'}), \right. \\ \left. \forall l, l' \in \mathcal{L}, \tau(l') = \eta(l) \right\}. \quad (29)$$

Note that Π is, in general, non-convex.

Proposition 6. The congestion-free set is a lower-set.

Proof: Consider $x \in \Pi$ and any $x' \in L(x)$. For all $l, l' \in \mathcal{L}, \tau(l') = \eta(l)$, we have $\min\{x'_l, \bar{q}_l\} \leq \min\{x_l, \bar{q}_l\}$ and $(c_{l'} - x_{l'}) \leq (c_{l'} - x'_{l'})$. Therefore, $\min\{x'_l, \bar{q}_l\} \leq \frac{\alpha_{l:l'}}{\beta_{l:l'}}(c_{l'} - x'_{l'})$. Thus $x' \in \Pi$, which indicates Π is a lower-set. ■

The predicate $(x \in \Pi)$ can be written as a Boolean logic formula over predicates in the form of (6) as:

$$(x \in \Pi) = \bigwedge_{l, l' \in \mathcal{L}, \tau(l') = \eta(l)} \left(\begin{aligned} & \left((x_l \leq \bar{q}_l) \wedge \left(x_l + \frac{\alpha_{l:l'}}{\beta_{l:l'}} x_{l'} \leq \frac{\alpha_{l:l'}}{\beta_{l:l'}} c_{l'} \right) \right) \vee \\ & \left(q_l + \frac{\alpha_{l:l'}}{\beta_{l:l'}} x_{l'} \leq \frac{\alpha_{l:l'}}{\beta_{l:l'}} c_{l'} \right) \end{aligned} \right). \quad (30)$$

Notice how the minimizer in (29) is translated to a disjunction in (30).

Now we explain the controls. The actuated flow of link l at time t is denoted by $\vec{q}_{l,t}$, where we have the following relations:

$$\vec{q}_{l,t} = \begin{cases} s_{l,t}q_{l,t}, & l \in \mathcal{L}_r, \\ \min\{q_{l,t}, r_{l,t}\}, & l \in \mathcal{L}_o, \\ q_{l,t}, & l \in \mathcal{L}_f, \end{cases} \quad (31)$$

where $s_{l,t} \in \{0, 1\}$ is the traffic light for link l , where 1 (respectively, 0) stands for green (respectively, red) light, and $r_{l,t} \in \mathbb{R}_+$ is the ramp meter input for on-ramp l at time t . The ramp meter input limits the number of vehicles that are allowed to enter the freeway in one time step. In order to disallow simultaneous green lights for links l, l' (which are typically pair of links pointing toward a common intersection in perpendicular directions), we add the additional constraints $s_{l,t} + s_{l',t} \leq 1$. In simple gridded networks, as in our case study network illustrated in Fig. 5, it is more convenient to define phases for actuation in north-south or east-west directions that are unambiguously mapped to the traffic lights for each individual link. The evolution of the network is given by:

$$x_{l,t+1} = x_{l,t} - \vec{q}_{l,t} + w_{l,t} + \sum_{l', \eta(l')=\tau(l)} \beta_{l':l} \vec{q}_{l',t}, \quad (32)$$

where $w_{l,t}$ is the number of exogenous vehicles entering link l at time t , which is viewed as the adversarial input. The evolution relation above can be compacted into the form (4):

$$x_{t+1} = f_{\text{traffic}}(x_t, u_t, w_t), \quad (33)$$

where u_t and w_t are the vector representations for control inputs (combination of traffic lights and ramp meters) and disturbances inputs, respectively. Note that f_{traffic} represents a hybrid system which dynamics of each mode is affine. The mode of the system is determined by the control inputs and state (which determines the minimizer arguments). Some works consider nonlinear representations for the fundamental digram (Fig. 4), but they still can be approximated using piecewise affine functions.

B. Monotonicity

Theorem 6. System (33) is monotone in Π .

Proof: Consider $x', x \in \Pi, x \preceq x'$. We need to show that $f_{\text{traffic}}(x, u, w) \preceq f_{\text{traffic}}(x', u, w), \forall w \in \mathcal{W}, \forall u \in \mathcal{U}$. We observe in (32) that all we need to verify is proving that $x_l - \vec{q}_l$ is a non-decreasing

function of x_l , since all other terms are additive and non-decreasing with respect to x . Since $x, x' \in \Pi$, the last argument in (28) is never the minimizer. Therefore, for all $l \in \mathcal{L}$, we have $x_l - \vec{q}_l \in \{0, x_l - r_l, x_l - c_l, x_l\}$, depending on the mode of the system and actuations, which all are non-decreasing functions of x_l . Thus, f_{traffic} is monotone in Π . ■

The primary objective in our traffic management approach is finding control policies such that the state is restricted to Π , which not only eliminates congestion, but also ensures that the system is monotone hence the methods of this paper become applicable. It is worth to note that the traffic system becomes non-monotone when flow is congested in diverging junctions, as shown in [51]. This phenomena is attributed to the first-in-first-out (FIFO) nature of the model. By assuming fully non-FIFO models, system becomes monotone in the whole state space. For a more thorough discussion on physical aspects of monotonicity in traffic networks, see [18].

The maximal system in (33) corresponds to the scenario where each w_l is equal to its maximum allowed value w_l^* . As mentioned earlier, it is reasonable to assume that the components of w_l are stochastically independent hence we have $\mathcal{W} = L(w^*)$.

C. Case Study

Network: Consider the network in Fig. 5, which consists of urban roads (links 1-26, 27,29,31,33 and 49-53), freeway segments (links 35-48) and freeway on-ramps (links 28,30,32,34). The layout of the network illustrates a freeway passing by an urban area, which is common in many realistic traffic layouts. There are 14 intersections (nodes a-n) controlled by traffic lights. Each intersection has two modes of actuation: north-south (NS) and east-west (EW). There are four entries to the freeway (nodes o-r) that are regulated by ramp meters. We have $n = 53$ and $\mathcal{U} = \mathbb{R}_+^4 \times \{0, 1\}^{14}$. Vehicles arrive from links 1,6,11,15,19,23,35,42,49 and 52. The parameters of the network are shown in Table I.

specification: As mentioned earlier, the primary objective is keeping the state in the congestion-free set. In addition, since the demand for the north-south side roads (links 49-53) is smaller than the traffic in the east-west roads, we add a timed liveness requirement for the traffic flow on links 49-53 as follows:

$$\phi = \bigwedge_{l=49,50,\dots,53} (x_l \geq 5) \Rightarrow \mathbf{F}_{[0,3]}(x_l \leq 5),$$

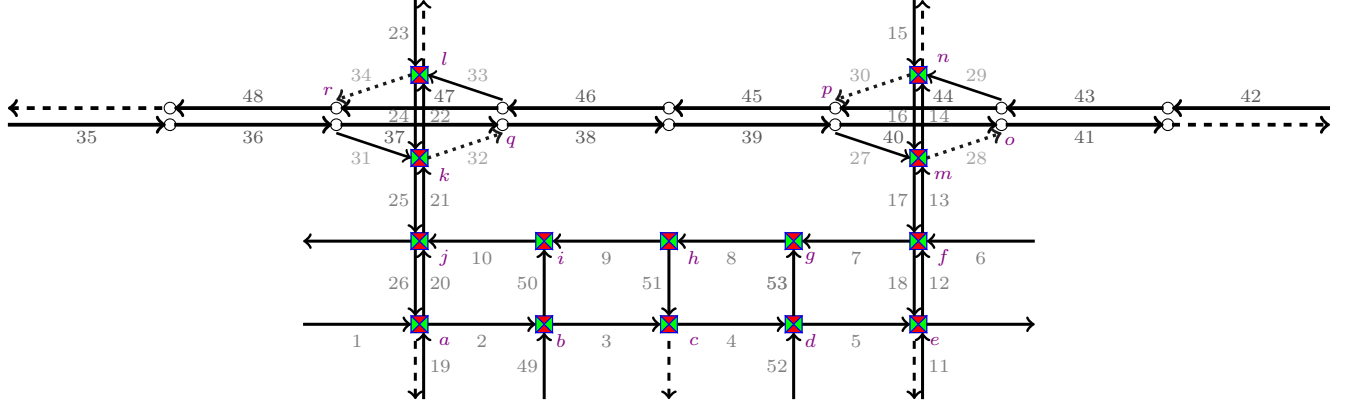


Fig. 5. Traffic management case study: A network of freeways and urban roads. There are 14 intersections controlled by traffic lights and 4 ramp meters.

TABLE I
PARAMETERS OF THE NETWORK IN FIG. 5

links	parameters
1 – 26, 49 – 53	$\bar{q}_l = 15, c_l = 40$
27 – 34	$\bar{q}_l = 15, c_l = 30$
35 – 48	$\bar{q}_l = 40, c_l = 60$
Turning ratios	value
$\beta_{2:50}, \beta_{4:53}, \beta_{8:51}, \beta_{12:7}, \beta_{13:28}, \beta_{15:30}, \beta_{16:28},$ $\beta_{21:32}, \beta_{24:32}, \beta_{26:2}, \beta_{36:31}, \beta_{36:33}, \beta_{39:27}, \beta_{43:29}$	0.2
$\beta_{5:12}, \beta_{6:13}, \beta_{6:18}, \beta_{10:21}, \beta_{10:26}$	0.3
$\beta_{1:20}, \beta_{6:7}$	0.4
$\beta_{1:2}, \beta_{11:12}, \beta_{14:30}, \beta_{17:7}, \beta_{17:18}, \beta_{19:2}, \beta_{19:20},$ $\beta_{22:34}, \beta_{23:24}, \beta_{23:34}, \beta_{27:14}, \beta_{27:17}, \beta_{29:16}, \beta_{31:22},$ $\beta_{31:25}, \beta_{33:24}, \beta_{49:3}, \beta_{49:50}, \beta_{51:4}, \beta_{52:5}, \beta_{52:53},$	0.5
$\beta_{2:3}, \beta_{3:4}, \beta_{4:5}, \beta_{8:9}, \beta_{12:13}, \beta_{13:14},$ $\beta_{15:16}, \beta_{16:17}, \beta_{20:21}, \beta_{21:22}, \beta_{24:25},$ $\beta_{25:26}, \beta_{36:37}, \beta_{39:40}, \beta_{43:44}, \beta_{46:47},$	0.8
Capacity ratios	value
$\alpha_{19:2}, \alpha_{26:2}, \alpha_{17:7}, \alpha_{12:7}, \alpha_{13:28}, \alpha_{16:28}$ $\alpha_{14:30}, \alpha_{15:30}, \alpha_{21:32}, \alpha_{24:32}, \alpha_{22:34}, \alpha_{23:34}$	0.5
Disturbances (arrival rates)	
$w_1^* = w_6^* = 4.5, w_{11}^* = w_{15}^* = w_{19}^* = 5, w_{23}^* = 6$ $w_{35}^* = w_{42}^* = 20, w_{49}^* = w_{52}^* = 2$	

which states that “if the number of vehicles on any of the north-south side roads exceeds 5, their flow is eventually actuated within three time units ahead”. The global specification is given as:

$$\mathbf{G}_{[0,\infty]}((x \in \Pi) \wedge \phi). \quad (34)$$

Note that $h^\varphi = 3$, $\varphi = (x \in \Pi) \wedge \phi$.

Open-loop Control Policy: We use Theorem 3. The shortest φ -sequence that we found for this problem has $T = 5$. The corresponding MILP had 2357 variables (of which 1061 were binary) and 4037 constraints ⁴, which is solved using the Gurobi MILP solver in 5.6 seconds on a dual core 3.0 GHz MacBook Pro. The cost is set to zero in order to just check for feasibility. Even though finding an optimal solution and checking for feasibility of a MILP have the same theoretical complexity, the latter is executed much faster in practice. A problem of this size (53 dimensional state) is virtually intractable using any method that involves state-space discretization. The control decisions in the resulting φ -sequence are shown in Table II. As stated in Theorem 3, starting from an initial condition in $L(x_0^{x_0, \varphi})$, applying the open-loop control policy (16) guarantees satisfaction of the specification. In other words, after applying the initialization segment, the repetitive controls in Table II become a fixed time-table for the inputs of the traffic lights and the ramp meters. Starting from x_0^φ , which is a 53-dimensional vector, we apply (16) using the values in Table II. The trajectory of the maximal system is shown in Fig. 6 (a). The traffic signals are coordinated such that the traffic flows free of congestion. The black dashed lines represent the capacity of the links, and the dashed line in the fourth figure (from the left) represents the threshold for the liveness sub-specification (ϕ). It is observed that all the state values for side road links (49-53) persistently fall below the threshold. The robustness values for $(x \in \Pi)$ and ϕ are shown in the fifth figure. As mentioned earlier, robustness corresponds to the minimum volume of vehicles that the system is away from congestion, or violating the specification. As expected by the correctness properties, the robustness values are always positive, indicating satisfaction.

As stated in Theorem 4, the trajectory of the maximal system converges to a periodic orbit. It is worth to not that the number of vehicles on freeway links is significantly smaller than its capacity, which is attributed to the fact that the number designated for \bar{q} (related to the maximum speed) of freeway links is relatively large (30, as opposed to 15 for roads). Therefore, freeway links are utilized in a way that there is enough space for high speed non-congested flow.

Robust MPC: Here it is assumed that the controller has full state knowledge. We apply the techniques developed in Sec. VI. Using the result from the previous section, the set $\Omega_{\mathcal{L}^\varphi}$ is constructed in \mathbb{R}_+^{212} . The cost criteria that we use in this case study is the total delay induced

⁴The scripts for this case study are available in <http://blogs.bu.edu/sadra/format-monotone>

TABLE II
 φ -SEQUENCE IN THE CASE STUDY

-	Initialization			Repetitive Controls				
node	u_0^φ	u_1^φ	u_2^φ	u_3^φ	u_4^φ	u_5^φ	u_6^φ	u_7^φ
<i>a</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>
<i>b</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>
<i>c</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>
<i>d</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>
<i>e</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>
<i>f</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>
<i>g</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>
<i>h</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>
<i>i</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>
<i>j</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>
<i>k</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>
<i>l</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>EW</i>
<i>m</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>
<i>n</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>EW</i>	<i>NS</i>	<i>NS</i>	<i>EW</i>
<i>o</i>	0.0	0.0	0.0	12.8	0.0	0.0	0.0	0.0
<i>p</i>	4.0	14.0	0.0	9.5	0.0	4.0	11.5	0.0
<i>q</i>	0.0	0.0	10.0	0.0	2.5	0.0	0.0	10.0
<i>r</i>	5.5	0.0	4.0	14.0	11.5	5.5	0.0	4.0

in the network over the planning horizon H . A vehicle is delayed by one time unit if it can not flow out of a link in one time step, which may be because of the actuation (e.g., red light) or waiting for the flow of other vehicles in the same link (i.e., we have $x_l \geq c_l$). We are also interested in maximizing the STL robustness score. The cost function is:

$$J_{\text{traffic}}(x^H, u^H) = -\eta \rho(\mathbf{x}, \mathbf{G}_{[-h^\varphi+1, H-h^\varphi]} \varphi, t) + \sum_{k=0}^{H-1} \gamma^k \sum_{l \in \mathcal{L}} (x_{l,t+k} - \vec{q}_{l,t+k}), \quad (35)$$

where \vec{q}_l , given by (31), is the amount of vehicles that flow out of link l , γ is the discount factor for delays predicted in further future, and η is a positive weight for robustness. Notice the connection between the time window of STL robustness score in (35) and MPC constraint enforcement in (26). It follows from Theorem 6 and STL quantitative semantics (3) that the cost function above is non-decreasing with respect to the state in Π . Therefore, in order to minimize the worst case cost, the maximal system is considered in the MPC optimization problem.

Starting from $x_0 = 0$, we implement the MPC algorithm (27) with $H = 3$ for 40 time steps. We set $\eta = 1000$, $\gamma = 0.5$ in (35). The disturbances at each time step were randomly drawn from $L(w^*)$ using a uniform distribution. The maximum computation time for each MPC step time step was less than 0.8 seconds (less than 0.5 seconds on average). The resulting trajectory is shown in Fig. 6 (b). For the same sequence of disturbances, the trajectory resulted from applying the open-loop control policy (16) (using the values in Table II) is shown in Fig. 6 (c). Both trajectories satisfy the specification. However, robust MPC has obviously better performance when costs are considered. The total delay accumulated over 40 time steps is:

$$J_{40} = \sum_{\tau=0}^{40} \sum_{l \in \mathcal{L}} (x_{l,\tau} - \vec{q}_{l,\tau}).$$

The cost above obtained from applying robust MPC was $J_{40} = 1843$, while the one for the open-loop control policy was $J_{40} = 2299$, which demonstrates the usefulness of the state knowledge in planning controls in a more optimal way. An optimal tuning of parameters η and γ requires an experimental study which is out of scope of this paper. We only remark that we obtained larger delays with non-zero η , which shows that including STL robustness score can be useful even though the ultimate goal is minimizing the total delay.

It is worth to note that we also tried implementing the MPC algorithm (for the case $\mathbf{w} = (w^*)^\omega$, or the maximal system) without the terminal constraints, as in (25). The MPC got infeasible at $t = 8$. The violating constraints were those in $x \in \Pi$. This observation indicates that the myopic behavior of MPC in (25), when no additional constraints are considered, can lead to congestion in the network.

VIII. CONCLUSION AND FUTURE WORK

In this paper, we provided a framework for control synthesis for positive monotone discrete-time systems from STL specifications. We showed that open-loop control sequences are sufficient and (almost) necessary for guaranteeing the correctness of STL specifications. We also developed a robust MPC method to plan controls optimally, while guaranteeing infinite time global STL specifications. We demonstrated the usefulness of our results on traffic management.

Future work will focus on non-monotone systems with parametric uncertainty whose state evolution can be over-approximated in an appropriate way using monotone systems. We will develop adaptive control schemes to tune parameters automatically using the data gathered from

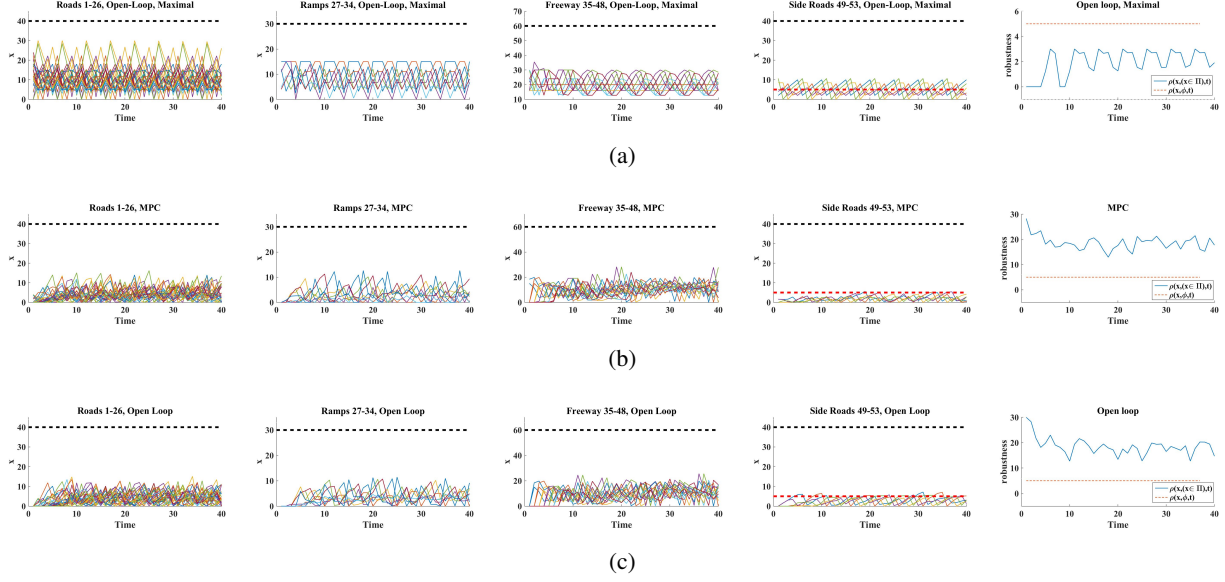


Fig. 6. Traffic management case study: (a) the trajectory of the maximal system obtained from applying the open-loop control policy (16) with initial condition x_0^φ (b) robust MPC generated trajectory with zero initial condition with disturbances chosen uniformly $L(w^*)$ (c) trajectory generated from applying the open-loop control policy (16) with zero initial conditions and the same disturbances as in (b)

the evolution of the system. This will eventually lead to data-driven control techniques for transportation networks with formal guarantees.

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